Scale-Free properties of weighted random graphs: Minimum Spanning Trees and Percolation

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Abstract. We study Erdös-Rényi random graphs with random weights associated with each link. In our approach, nodes connected by links having weights below the percolation threshold form clusters, and each cluster merges into a single node, thus generating a new "clusters network". We show that this network is scale-free with $\lambda = 2.5$. Furthermore, we show that optimization causes the percolation threshold to emerge spontaneously, thus creating naturally a scale-free "clusters network". This phenomenon may be related to the evolution of several real world scale-free networks.

Our results imply that: (i) the minimum spanning tree (MST) in random graphs is composed of percolation clusters, which are interconnected by a set of links that create a scale-free tree with $\lambda = 2.5$ (ii) the optimal path may be partitioned into segments that follow the percolation clusters, and the lengths of these segments grow exponentially with the number of clusters that are crossed (iii) the optimal path in scale-free networks with $\lambda < 3$ scales as $\ell_{opt} \sim \log N$, and the weights along the optimal path decay exponentially with their rank.

Keywords: minimum spanning tree, percolation, scale-free, optimization
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INTRODUCTION

Scale-free topology is very common in natural and man-made networks. Examples vary from biological networks (proteins), social contacts between humans, and technological networks such as the World Wide Web or the Internet [1, 2, 3]. Scale free (SF) networks are characterized by a power law distribution of connectivities $P(k) \sim k^{-\lambda}$, where k is the degree of a node (i.e. the number of nearest neighbors connected to it) and λ controls the broadness of the distribution. Many of these networks are observed to have typical values of λ around 2.5. For values of $\lambda < 3$ the second moment of the distribution $\langle k^2 \rangle$ diverges, leading to several anomalous properties of these networks. For example [4, 5]: they are robust to random breakdown of links or nodes, and their radius scales as $l_{min} \sim loglogN$.

In many real world networks there is a "cost" or a "weight" associated with each link, and the larger the weight on a link, the harder it is to traverse this link. In this case, the network is called "weighted" [6]. Examples can be found in communication and computer networks, where the weights represent the bandwidth or delay time, in protein networks where the weights can be defined by the strength of interaction between

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proteins [7, 8] or their structural similarity [9], and in sociology where the weights can be chosen to represent the strength of a relationship [10, 11].

In this paper we suggest a simple process that generates random scale-free networks with $\lambda = 2.5$ from weighted Erdös-Rényi graphs [12]. This is performed by merging all nodes connected by links with weights below the percolation threshold into a single node, and thus generating a new scale-free "clusters network". We further show that the minimum spanning tree (MST) on an Erdös-Rényi graph is composed of percolation clusters interconnected by a scale-free tree, and use percolation theory to study the expected weights along a random path on this tree. Using these results we study the structure of the optimal path in ER and scale-free graphs – in particular we explain the observed fact that the optimal path in scale-free networks with $\lambda < 3$ scales as $\ell_{opt} \sim \log N$ (i.e. the ultra-small world property is lost), and show that the weights along the optimal path decay exponentially with their rank.

THE CLUSTERS NETWORK:

Consider an Erdös-Rényi (ER) graph with N nodes and an average degree $\langle k \rangle$, with a total of $\frac{1}{2}N\langle k \rangle$ links. We distribute weights on the links, chosen randomly and uniformly from the range $[0,1]$. It is known that in ER graphs the critical percolation probability is $p_c = \frac{1}{\langle k \rangle}$ [12]. If we distinguish between links with weight above and below p_c, the weights below p_c will create percolation clusters at criticality. From percolation theory [13] it follows that these clusters are distributed according to a power law: $n_s \sim s^{-\tau}$, where n_s is the number of clusters of size s, and $\tau = 2.5$ for infinite dimension [1]. Next, we merge all nodes inside each cluster into a single node [2]. We define a new "clusters network" of N_{CL} nodes which consists of these clusters. We consider again all links of the original network, and each link with a weight *larger* than p_c is considered as a link between these new nodes, see Fig. (1a).

The degree distribution of the clusters network is: $P(k) \sim k^{-\lambda}$, with $\lambda \simeq 2.5$, as can be seen from simulations in Fig. (2). The behavior of $P(k)$ can be understood if we assume that the degree k of each node in the clusters network is proportional to the cluster size s, which is also distributed according to the same power-law $n_s \sim s^{-2.5}$. We further studied the "tomography" [14, 15] of the clusters network, i.e. the number of nodes N_ℓ at each chemical distance ℓ from the most connected node, and found that it does not change when we randomize the links [16] in the network. This indicates that the clusters network is random.

The number of nodes N_{CL} in the clusters network is actually the number of clusters in an Erdös-Rényi graph at the percolation threshold p_c. This quantity, which is non-universal, was found for various two and three dimensional percolating systems [17, 18]. For ER graphs we have (see [19] and Appendix A) $N_{CL} = \frac{N}{2}$.

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[1] ER networks can be regarded as having an infinite dimension since space does not play any role.
[2] This is done by the following algorithm: we follow all links of the network. If there exists a link i between any two nodes with weight smaller than $p_c = 1/\langle k \rangle$, we identify the nodes as belonging to the same cluster.

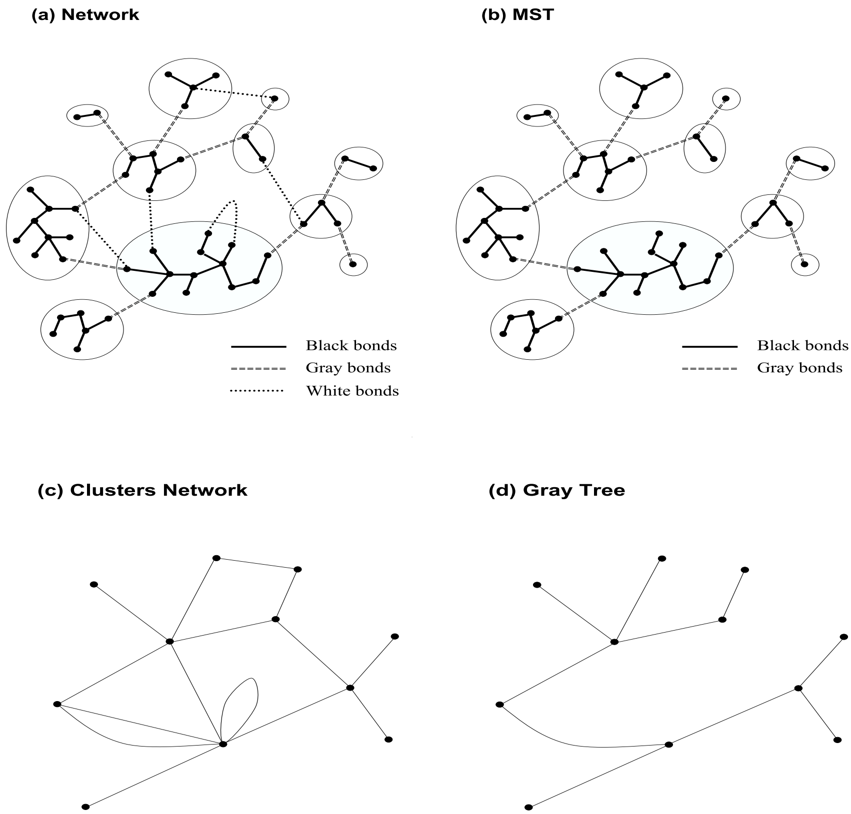

FIGURE 1. Sketch of the "clusters network". (a) The original ER network, partitioned into percolation clusters whose sizes s are power-law distributed: $n_s \sim s^{-\tau}$ ($\tau = 2.5$ for ER graphs). The "black" links are the links with weights below p_c, the "white" links are the links that are removed by the bombing algorithm, and the "gray" links are the links whose removal will disconnect the network (and therefore are not removed even though their weight is above p_c). (b) The minimum spanning tree (MST), composed of black and gray links only. (c) The "clusters network". The nodes are the clusters in the original network and the links are the links connecting nodes in different clusters (i.e. "white" and "gray" links). The clusters network is scale-free with $\lambda = 2.5$. Notice the existence of double connections and self-loops. (d) The "gray tree", which is reached by bombing the clusters network (by removing the "white" links). The MST is composed of the gray tree, which is scale-free, interconnecting the percolation clusters.

MINIMUM SPANNING TREES:

We next show that the minimum spanning tree (MST) on an ER graph is related to the clusters network and thus exhibits scale-free properties. The MST on a weighted graph is a tree that reaches all nodes of the graph, and whose total weight is minimal.

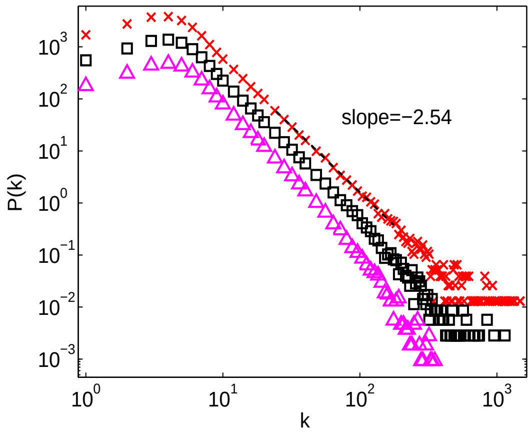

FIGURE 2. The degree distribution of the clusters network, where the new nodes are the percolation clusters and the new links are the links with weights larger than p_c (x). The distribution exhibits a scale-free tail with $\lambda \simeq 2.5$. If we assume in our process a threshold less than p_c we obtain the same power law degree distribution with an exponential cutoff. The different symbols represent different threshold values: $p = p_c - 0.03$ (□) and $p = p_c - 0.05$ (△). The original ER network has $N = 50000$ and $\langle k \rangle = 5$. Note that around $\langle k \rangle$ the degree distribution has a maximum.

Each path between two sites on the tree is the optimal path in the "strong disorder" limit, meaning that it is the path along which the maximum barrier is minimal [19, 20, 21]. Standard algorithms for finding the MST [22] are Prim's algorithm, which resembles invasion percolation, and Kruskal's algorithm, which resembles percolation. An equivalent algorithm to find the MST is the bombing optimization algorithm [20, 21]: First we mark all the links of the graph as "black". We then remove the links in order of descending weights. If the removal of a link disconnects the graph, we restore the link and mark it as "gray" [23], otherwise - we mark it "white". The algorithm ends when no more links can be removed without disconnecting the graph.

In this algorithm, only links that close a loop can be removed. Thus, the bombing does not modify the percolation clusters – where the links have weights below p_c – because below criticality the presence of loops is negligible [12, 13]. In fact, the bombing modifies only links *outside* the clusters, so actually it is the links of the *clusters network* which are bombed. Hence the MST is composed of percolation clusters (i.e. nodes connected by links with weights below p_c) connected by gray links, as seen in Fig. (1b).

We now generate a new tree from the MST, whose nodes are the clusters and whose links are the gray links connecting them. We call this tree the "gray tree" (see Fig. 1d). Note that bombing the original ER network to obtain the MST is equivalent to bombing the clusters network (which is scale-free) to obtain the gray tree, because the links inside the clusters effectively do not participate in the bombing. The gray tree is found to have a scale-free degree distribution with $\lambda \simeq 2.5$ – same as the clusters network –

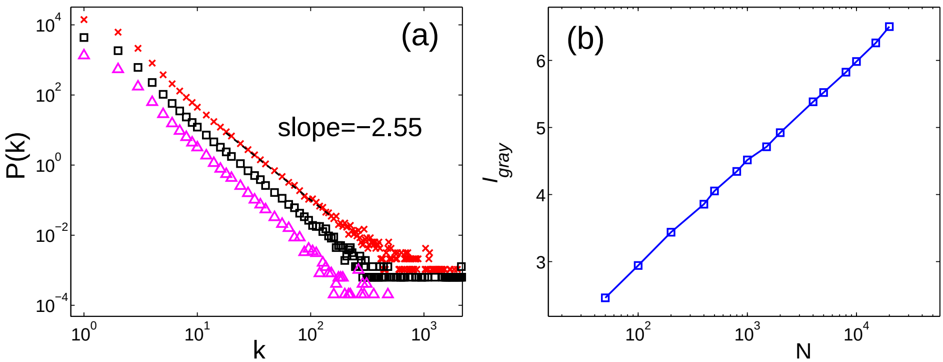

FIGURE 3. (a) The degree distribution of the "gray-tree",where the new nodes are the percolation clusters and the new links are the gray links. This is actually an MST on the clusters network. Different symbols represent different threshold values: p_c (x), $p_c + 0.01$ (□) and $p_c + 0.02$ (△). The distribution exhibits a scale-free tail with $\lambda \simeq 2.5$, and is relatively not sensitive to changes in p_c. (b) The average path length on a the gray-tree as a function of original network size. It is seen that $\ell_{gray} \sim \log N$.

as shown in Fig (3a) [3]. The average path length ℓ_{gray} is found to scale with N_{CL} as $\ell_{gray} \sim \log N_{CL} \sim \log N$, [4] as shown in Fig. (3b). Note that even though the gray tree is scale-free, it is not ultra-small [5].

To summarize, the Minimum Spanning Tree is composed of clusters of "black links" - with weights below the critical probability p_c, which are connected by a scale-free tree of "gray links" - whose weights are above p_c. The properties of the percolation clusters at the critical threshold are well known, for example: they are fractal, their average degree is $\langle k \rangle = 1$, and their average length is $l \sim N^{1/3}$ [21]. On the other hand, the gray tree is well above criticality (because the gray links were bombed and then restored), and $\ell_{gray} \sim \log N$. In the following section we will focus on other properties of the gray links.

STRUCTURE OF OPTIMAL PATH

The *optimal path* between any two points on a weighted random graph is the path with minimal barrier between them [5]. Many physical systems follow the optimal path in phase space, which represents the trajectory with minimal energy barrier. It can be shown that this path lies on the MST [20], and thus it can be decomposed into segments along the percolation clusters (black links), and "crossings" with weights above p_c (gray links)

[3] MST's on scale-free networks were observed to retain the original network's degree distribution [24, 25].
[4] Although Braunstein et. al [21] found that the length of the optimal path $\ell_{opt} \sim \log^{\lambda-1} N$ for SF networks with $\lambda < 3$ in the strong disorder limit, this is valid only when there do not exist multiple links between nodes. For SF networks that do not have this restriction, such as in our case, we find a shorter optimal path: $\ell_{opt} \sim \log N$ for $\lambda < 3$.
[5] This path is also referred to as the "min-max" path.

connecting these segments – see Fig (1b).

Accordingly, we study the properties of paths between all pairs of nodes on the MST. On each path we order the weights by rank, and average the weight of rank 1, rank 2, etc. over all paths – see Fig. (4). In the figure we can clearly distinguish between the black and gray links. The black links consist the linear regime, because they had no loops in the original network, and therefore they were not affected by the bombing. Consequently, they are uniformly distributed and their number scales as $N^{1/3}$. As opposed to this, the gray links were bombed and then restored. Therefore they have a non-uniform distribution, and their number scales logarithmically with N.

Close examination of the gray links in Fig (4) reveals that for small systems the linear regime begins actually *above* p_c. This may be explained by the fact that the percolation threshold for small systems is $p_c + \Delta p_c$ because of finite size effects [6] – see Appendix B. For random graphs we have $\Delta p_c = \frac{p_c}{\ell_{opt}}$, where $\ell_{opt} \sim N^{\nu_{opt}}$ is the average length of the percolation cluster, and for ER graphs $\nu_{opt} = \frac{1}{3}$ [21]. Moreover, we observe that the gray links from optimal paths of small systems are found in large systems also, and that new gray links are added only in the fluctuation region $[p_c, p_c + \Delta p_c]$. For example: the largest weights $w_{max} \approx 0.34$ and $w_2 \approx 0.25$ appear in system of all sizes, and as the system size grows, new gray links are added with smaller weights only. Thus, long optimal paths (i.e., optimal paths in large systems) include all gray links of short optimal paths. In this sense, large MST's "grow" from small MST's: the small system can be found inside the large system, and thus it is reasonable to say that there is a self-similarity between large and small systems. [7]

These observations suggest the following explanation (see Fig. (5)): For a random graph of size N the optimal path follows percolation clusters and crosses between them along gray links. Nevertheless, it always includes the infinite percolation cluster, which is the largest cluster of the system. A path which spans the percolation cluster must account for weights up to $p_c + \frac{p_c}{\ell_{opt}}$ (where $\ell_{opt} \sim N^{1/3}$ in ER graphs) in order to cross out of it (see Appendix B). This is actually the weight of the gray link associated with the largest percolation cluster, which is by definition the minimal weight link required by the optimal path in order to cross from this cluster to its neighboring clusters. Thus, the optimal path, which is the path with minimum barriers, follows the gray links which are determined by the fluctuations in p_c along different clusters.

Because of the self-similarity, the optimal path in large systems includes all clusters and gray links of the optimal path in small systems. It is also known that the optimal path is dominated by the length of the infinite percolation cluster [8]. As noted before, we have observed that the number of crossings grows logarithmically with system size. This is because for a new cluster to be added to the optimal path - its length must grow by an

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[6] The quantity Δp_c is sometimes referred to as "the fluctuations in critical probability".

[7] The links of highest weight can be associated with gray links outgoing from very small clusters. These cannot be optimized (due to limited number of exits) and therefore do not change with the network size. In an ER graph of average degree $\langle k \rangle$, the smallest cluster consists of a single node with $\langle k \rangle$ links all above p_c. The minimal of these $\langle k \rangle$ links is the largest gray link, and thus we have: $\langle w_{max} \rangle = p_c + \frac{1-p_c}{\langle k \rangle + 1}$.

[8] Both the optimal path length ℓ_{opt} and the length of the infinite percolation cluster scale as $N^{\nu_{opt}}$, meaning that there is a constant proportion between them for sufficiently large N [21].

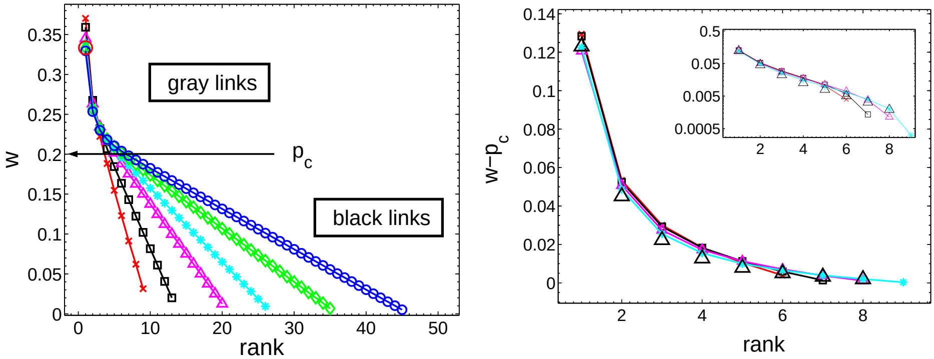

FIGURE 4. (left) Weights along the optimal path on an ER graph with $\langle k \rangle = 5$, sorted according to their rank. Weights below $p_c = 0.2$ are the "black" links which are uniformly distributed, while the weights above p_c - the "gray" links are not. Different symbols correspond to different graph sizes: $N = 100$ (x), $N = 200$ (□), $N = 500$ (△), $N = 1000$ (∗), $N = 2000$ (⋄), and $N = 4000$ (○). Note that for finite N the linear regime starts *above* p_c because of the fluctuations in the critical probability $\Delta p_c \sim \frac{p_c}{\ell_{opt}}$. Note also that each path includes the gray links of shorter paths.
(Right) The gray links for graphs of different sizes. Triangles (△) represent $\frac{p_c}{\ell_{opt}}$ where ℓ_{opt} is the measured optimal path length at which we encounter this weight (see Fig. 5). ℓ_{opt} grows exponentially with ℓ_{gray} leading to an exponential decay of the weights with their rank (inset).

order of magnitude: $l \rightarrow l + o(l) = l + \alpha l = \beta l$ (where $\beta > 1$). Thus the growth rate of the crossings with system size l is inversely proportional to the system size:

$$\frac{\Delta l_g}{\Delta l} = \frac{1}{l} \Rightarrow l_g \sim \log l \sim \log N \tag{1}$$

where $l \equiv \ell_{opt} \sim N^{1/3}$.

From Fig. (4) we can see that the weights along the optimal path decay exponentially as a function of their rank. This is because each time a new cluster is added to the optimal path, its size is of the order of the system size ℓ_{opt}, and it consists of links below $\tilde{p_c} = p_c + \frac{p_c}{\ell_{opt}}$ (see Fig. (5)). Thus the new gray link which connects the optimal path to the new cluster is of weight $\tilde{p_c} = p_c + \frac{p_c}{\ell_{opt}}$. The weights of the crossings are: $p_c + \frac{p_c}{l_0}, p_c + \frac{p_c}{\beta l_0}, p_c + \frac{p_c}{\beta^2 l_0}, p_c + \frac{p_c}{\beta^3 l_0}, ..., p_c + \frac{p_c}{\beta^r l_0}$. which gives an exponential decay.

DISCUSSION, SUMMARY AND CONCLUSIONS

We have introduced a method for treating scale-free (SF) networks with $\lambda = 2.5$ by considering the "clusters network" of weighted Erdös-Rényi (ER) graphs. This method can be generalized if we take the original network to be a weighted scale-free graph with $3 < \lambda < 4$ rather than ER. In this case the cluster size distribution is [26] $n_s \sim s^{-\tau}$ where $\tau = \frac{2\lambda - 3}{\lambda - 2} \in [2.5, 3]$, and thus the clusters network is scale-free with a distribution of $P(k) \sim k^{-\tilde{\lambda}}$ with $2.5 \leq \tilde{\lambda} \leq 3$. Notice also that there is a correspondence between the

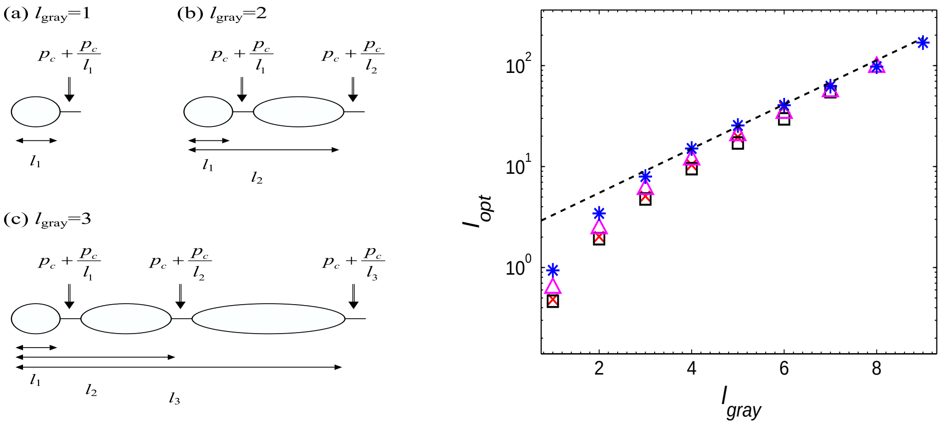

FIGURE 5. (Left) An illustration of the optimal path. The optimal path follows the percolation clusters up to the largest cluster, crossing between clusters through the "gray" links which are the minimal barriers between the clusters. Because of the self-similarity, large paths include all the clusters and gray links of small paths, and the optimal path length is multiplied by a constant factor β each time a new cluster is added to it (i.e. $\ell_2 = \ell_1\beta$, $\ell_3 = \ell_2\beta = \ell_1\beta^2$, etc.). The weight of a gray link connecting a cluster to the optimal path is determined by the fluctuations in the critical threshold: $p_c + \frac{p_c}{\ell}$ where ℓ is the system size. (right) The optimal path length for systems of different size N, vs. the number of gray links encountered along the way. Different symbols represent different graph sizes: $N = 4000$ (x), $N = 8000$ ($\square$), $N = 32000$ ($\triangle$), $N = 128000$ ($*$). This figure shows that the segments of the optimal path inside percolation clusters grow exponentially with the number of gray links, and that there is a self-similarity between large and small systems (see left figure). Dashed line represents the curve $\ell_{opt} = A \cdot \beta^{\ell_{gray}}$ where $\beta \approx 1.66$ and $A = 2$.

TABLE 1. A comparison between the original graph and the corresponding clusters network for ER and SF networks. Notice that the fluctuations of p_c in the original network correspond to the critical probability in the clusters network.

original network	**clusters network**
Erdös-Rényi $p_c = \frac{1}{\langle k \rangle}$ $\Delta p_c \sim N^{-\nu_{opt}} = N^{-1/3} \to 0$	Scale-Free $\lambda = 2.5$ $p_c \sim N^{(\lambda-3)/(\lambda-1)} = N^{-1/3} \to 0$
Scale-Free $3 \le \lambda \le 4$ $\Delta p_c \sim N^{-(\lambda-3)/(\lambda-1)} \to 0$	Scale-Free $\tilde{\lambda} = \frac{2\lambda-3}{\lambda-2} \in [2.5, 3]$ $p_c \sim N^{(\tilde{\lambda}-3)/(\tilde{\lambda}-1)} \to 0$

fluctuations in p_c of the original network and the critical threshold $\tilde{p_c}$ of the clusters network. Table 1 gives a comparison between the original network and the clusters network for ER graphs and SF graphs.

To summarize, we have shown that any weighted random network hides an inherent scale-free network – the "clusters network" [9]. We have shown that the minimum spanning tree is built by "bombing" the clusters network, and thus it is composed of percolation clusters connected by a scale-free tree of "gray" links. We have also studied the optimal path, which is the average path along the MST. We have shown that the optimal path may be partitioned into segments that follow the percolation clusters, and the lengths of these segments grow exponentially with the number of clusters that are crossed. We have used the above results to show that the optimal path in scale-free networks with $\lambda = 2.5$ scales as $\ell_{opt} \sim \log N$, and that the weights along the optimal path decay exponentially with their rank.

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APPENDIX A: DERIVATION OF THE NUMBER OF CLUSTERS AT CRITICALITY IN RANDOM GRAPHS

In [19] the number of clusters at criticality for random graphs was generally found to be:

$$N_{CL} = N\left(1 - \frac{\langle k \rangle p_c}{2}\right). \tag{2}$$

This is because the MST consists of black links and gray links. The number of gray links is found by reducing the number of black links remaining at the percolation threshold, $\frac{1}{2}N\langle k \rangle \cdot p_c$, from the total number of links on the MST, which is equal to $N-1$. Using the fact that the number of clusters at criticality equals the number of the gray links plus one, we get equation (2). For Erdös-Rényi graphs we have: $p_c = \frac{1}{\langle k \rangle}$, hence: $N_{CL} = \frac{N}{2}$.

For graphs of finite size N we find a correction of the form: $n(N) = n(\infty) + \frac{1}{N}$, where $n(N) = \frac{N_{CL}}{N}$ is the number of clusters per site at criticality in a graph of size N, and $n(\infty) = \frac{1}{2}$ – see Fig. (6). This correction conforms with results found for percolation on lattices in finite dimension [18].

[9] Similar results can also be obtained for graphs embedded in two or three dimensions, with different power law exponents.

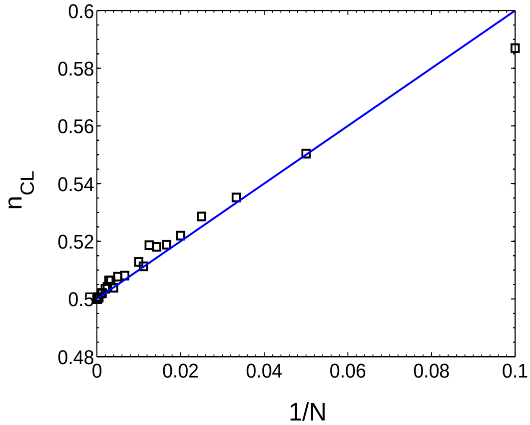

FIGURE 6. The number of clusters per site at the critical percolation threshold for ER graphs with $\langle k\rangle = 5$, as a function of graph size. The solid line represents the curve $n_{CL} = \frac{1}{2} + \frac{1}{N}$. For $N \to \infty$ we have $n_{CL} = \frac{1}{2}$.

APPENDIX B: THE FLUCTUATIONS IN THE CRITICAL PROBABILITY

It is known from percolation theory that the critical threshold for systems of finite size is not definite, rather it consists of a region $[p_c - \Delta p_c, p_c + \Delta p_c]$ in which the behavior is similar to criticality. In this region, with a high probability, there exist a cluster spanning the system and there is a negligible number of loops. The general form is: $\Delta p_c = \frac{p_c}{(L)^{1/\nu}}$ where L is the system length and ν is the correlation length exponent [27]. For random graphs [28]: $\Delta p_c = \frac{p_c}{\ell_{opt}}$ where $\ell_{opt} \sim N^{\nu_{opt}}$ is the average length of the percolation cluster, and $\nu_{opt} = \frac{1}{3}$ for ER graphs [21].

As a simple demonstration, consider the number of loops per node, which was found to be near criticality [29]:

$$\ell = \frac{N_{loops}}{N} \sim (p - p_c)^{\bar{\nu}} \tag{3}$$

where $\bar{\nu} = 3$ for ER graphs. Substituting $p = p_c + \frac{C}{N^{1/3}}$ (where $C = o(1)$) we get: $\frac{N_{loops}}{N} \sim \frac{C^3}{N}$, or $N_{loops} = o(1)$. Thus, the number of loops is negligible below $\tilde{p}_c = p_c + \Delta p_c$, where $\Delta p_c \sim \frac{1}{N^{1/3}}$.

In weighted ER graphs of size N we can say that a path spanning the percolation cluster (which is the largest loop-less sub-structure) is required to allow for weights smaller or equal to $\tilde{p}_c = p_c + \frac{p_c}{\ell_{opt}}$ where $\ell_{opt} \sim N^{1/3}$. This may be seen as a "resolution" problem, in which small systems do not have enough links to account for the exact p_c of the system.

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