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Universality classes for diffusion in the presence of correlated spatial disorder

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We study the moments of displacement of a diffusing particle in one dimension in the presence of correlated random fields. We find that even for short-range algebraic correlations there exists a critical moment q_c such that for $q<q_c$ the moments of displacement depend logarithmically on time, whereas for $q>q_c$ the moments of displacement are a power law of time. This result is surprising since short-range correlations do not usually change the scaling properties.

Recent years have witnessed increasing interest in diffusion in random media.[1-6] Some of the interesting structures over which diffusion takes place are topologically one dimensional. One example is diffusion on a polymer in a uniform external field, which generates a random field along the chain due to the random spatial structure of the polymer.[7] A second example is random impurities which produce *local* random fields that favor a given direction for the diffusing particle.

For the ideal case in which we may neglect the spatial correlations in the random fields, Sinai[3] solved exactly for the asymptotic time dependence of the mean-square displacement $\langle l^2\rangle$ *along the chain,*

$$\langle l^2\rangle \sim \ln^4 t\,. \tag{1}$$

The probability density $P(l,t)$ is, asymptotically, a normalized (with respect to l) function of the variable $u\equiv l/\ln^2 t$.[3] Nauenberg[8] (heuristically) and Kesten[9] (rigorously) have found that $P(l,t)$ is of the form $(\ln t)^{-2}\exp(-C|u|)$. Therefore, for the uncorrelated case, *all* positive moments scale identically; i.e., for $q>0$,

$$\langle |l|^q\rangle \sim (\ln t)^{2q} \quad \text{(uncorrelated)}\,. \tag{2}$$

Here, we discuss the more general case where the random fields are *correlated.* We present an exact proof that there exists a critical value $q=q_c$ above which $\langle |l|^q\rangle$ becomes a power law of t. We find that even for short-range algebraic correlations, Eq. (2) does not hold for *all moments.*

Consider a random walker on a given site j of the chain. The probabilities to proceed to the right and to the left are $W_{j,j+1}$ and $W_{j,j-1}$, respectively. Since their sum is one, we may write $W_{j,j\pm 1}=(1\pm E_j)/2$. We choose $E_j=|E|\tau_j$, where $\tau_j=\pm 1$. If the τ_j are correlated random variables, the random bias fields are correlated. This correlation in the τ_j is described by the correlation function of the Fourier components τ_k,

$$\langle \tau_k \tau_{-k}\rangle \sim \frac{1}{k^{\lambda}} \quad (k\ll 1)\,. \tag{3}$$

We choose to average[10] $\ln t$ for fixed l and find[6]

$$\langle \ln t\rangle \sim l^{(1+\lambda)/2}\,. \tag{4}$$

One way to generate a correlated configuration of τ_j is to choose segments or "strings" within which the sign is uniform (that uniform sign chosen randomly).[6] The probability to choose a string of length m is

$$P(m)\sim m^{-\beta} \tag{5a}$$

(analogous to "Lévy flight"). Here, $\beta>1$ (due to the normalization condition). One finds that λ depends on β, with[6]

$$\lambda=\begin{cases}1, & \beta\le 2\,,\\ 3-\beta, & 2\le\beta\le 3\,,\\ 0, & \beta\ge 3\,.\end{cases} \tag{5b}$$

Note that all correlations defined by $\beta\ge 3$ are algebraic decaying but short range. One might expect that for such short-range correlations, the Sinai result (2) for the uncorrelated limit would hold. This is not true, as we shall now prove.

The average $\langle |l|^q \rangle$ is actually a double average:

$$\langle |l|^q \rangle \equiv \langle \langle |l|^q \rangle_w \rangle_c , \tag{6}$$

where $\langle \ \rangle_w$ denotes the average over all possible walks, and $\langle \ \rangle_c$ denotes the average over all configurations. Thus, we average first over all possible walks for a given configuration of the random fields, and then we average over the configurations themselves.

The proof proceeds as follows. First we consider a configuration in which, from the initial site of the walker, there is a string of length m to its right in which $\tau_j = +1$, where j indexes the sites on the segment. A lower bound on the $\langle |l|^q \rangle_w$ for this configuration is given by

$$\langle |l|^q \rangle_w \geq \langle |l|^q \rangle_{w'(m)} , \tag{7}$$

where w' represents an average over walks in a system with a trap at the end of the segment such that a walk which reaches the end of the segment is frozen there. We observe that

$$\langle |l|^q \rangle_{w'(m)} \sim \begin{cases} t^q, & t \leq m , \\ m^q, & t > m . \end{cases} \tag{8}$$

The behavior for $t \leq m$ follows from the fact that the average displacement in the presence of a uniform field is proportional to time, while the behavior for $t > m$ follows from the fact that the walker is trapped at m once it arrives there. Since $P(m)$ is the probability that the initial string will be of length m,

$$\langle |l|^q \rangle \equiv \langle \langle |l|^q \rangle_w \rangle_c \geq \sum_m P(m) \langle |l|^q \rangle_{w'(m)} . \tag{9a}$$

From (8),

$$\langle |l|^q \rangle \geq \sum_{m<t} P(m) m^q + \sum_{m>t} P(m) t^q \sim t^{q-\beta+1}, \quad \text{for } q-\beta+1>0 . \tag{9b}$$

Since $\langle |l|^q \rangle$ is bounded from above by t^q, it follows that $\langle |l|^q \rangle$ behaves asymptotically as a power of t, for all values of β provided that $q > q_c$, where

$$q_c = \beta - 1. \tag{10}$$

We see now that even for the case $\lambda = 0$, i.e., $\beta > 3$ as long as β is finite, Eq. (2) is not valid for all moments.

We checked (9b) numerically for $\beta = 2.5$. In Fig. 1 we present simulation results for $\langle |l|^q \rangle$ for several values of q. For $q < q_c$ we find logarithmic behavior (Fig. 1), whereas for $q > q_c$ we find a power-law dependence in time (Fig. 2).

It is interesting to estimate the number of field configurations needed in a numerical simulation to see the power-law behavior predicted by Eq. (9b). The field configurations contributing to the second term on the right-hand side of Eq. (9b) are very rare. If we consider $\langle |l|^q \rangle_w$ for a *typical* configuration,[11] we find

$$\langle |l|^q \rangle_{\text{typ}} \sim (\ln t)^{2q/(1+\lambda)} . \tag{11}$$

The rare configurations are those where at time t the walker is still in the region of the first segment. The probability for such a configuration is

$$P_r(t) = \sum_{l>t} P(l) \sim (\beta-1) t^{-\beta+1} . \tag{12}$$

Note that the probability for a rare configuration depends on the number of time steps one takes. The number of configurations $\mathcal{N}_c$ needed to obtain power-law behavior with probability of order $\frac{1}{2}$ is obtained by the requirement that the average number of rare configurations is one out of $\mathcal{N}_c$, i.e., $P_r(t) = 1/\mathcal{N}_c$. Hence, the number of required configurations $\mathcal{N}_c$ is

$$\mathcal{N}_c = \frac{t^{\beta-1}}{\beta-1} . \tag{13}$$

We expect to observe a crossover in the time dependence of $\langle |l|^q \rangle$ (for $q > q_c$), from a power-law behavior in t (at small t) to a power law in $\ln t$ (at larger t) because in the finite sample of configurations we expect to find no rare configurations when the average number of rare configurations is smaller than one. From (13) it follows that the crossover time t^* scales with the number of

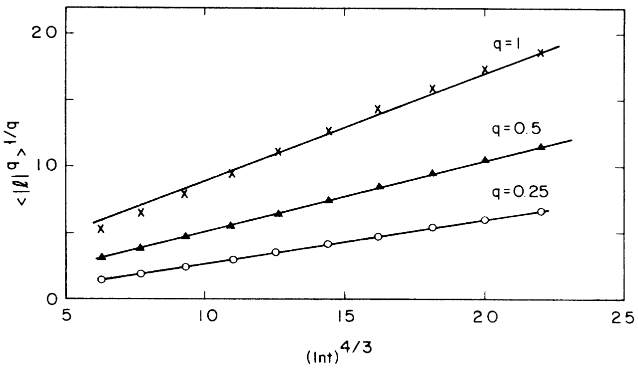

FIG. 1. Linear plot of $\langle |l|^q \rangle^{1/q}$ as a function of $(\ln t)^{4/3}$ with $\beta = 2.5$ ($\lambda = 3 - \beta = \frac{1}{2}$), for $q < q_c = \frac{3}{2}$. The values of q are 0.25 (o), 0.5 (▲), and 1 (×).

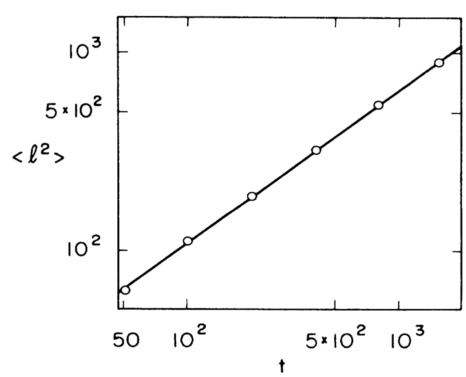

FIG. 2. Log-log plot of $\langle |l|^q \rangle$ for $q_c < q = 2$ as a function of t. The straight lines obtained here and in Fig. 1 support the crossover from logarithmic to power-law behavior predicted by Eq. (9b). Note that the slope, 0.75, falls in the predicted range between $q-\beta+1$ ($=0.5$) and q ($=2.0$).

configurations $\mathcal{N}$ over which we average

$$t^* \sim [(\beta-1)\mathcal{N}]^{1/(\beta-1)} . \qquad (14)$$

Hence, although we expect a power-law behavior of $\langle l^q \rangle$ above q_c at large times, this behavior is difficult to observe for the short-range ($\beta \gtrsim 3$) case in computer experiments because of the large number of configurations necessary.

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[10] Equation (4) in the $\lambda=0$ limit at first sight resembles the Sinai result for the uncorrelated limit. However, Sinai's result refers to moments of l for fixed t, whereas Eq. (4) refers to moments of $\ln t$ for fixed l.
[11] In order to understand the derivation of (11), note that in Ref. 6 the mean logarithm of the time to reach a distance L was found to scale as $\langle \ln t \rangle \sim L^{(1+\lambda)/2}$. The average over $\ln t$ corresponds to an average over *typical* configurations (see Ref. 6) so (11) follows.