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Anomalous ballistic diffusion

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We introduce a novel two-component random network. Unit resistors are placed at random along the bonds of a pure superconducting linear chain, with the distance l between successive resistors being chosen from the distribution $P(l) \sim l^{-(\alpha+1)}$ where $\alpha > 0$ is a tunable parameter. We study the transport exponents d_w and $\tilde{\zeta}$ defined by $\langle x^2 \rangle \sim t^{2/d_w}$ and $\rho \sim L^{\tilde{\zeta}}$, where $\langle x^2 \rangle$ is the mean-square displacement, ρ the resistivity, and L the system size. We find that for $\alpha \geq 1$ both d_w and $\tilde{\zeta}$ stick at their value for a nonzero concentration of resistors. For $\alpha < 1$ they vary continuously with α: $d_w = 2\alpha$ and $\tilde{\zeta} = \alpha$. In the presence of a *bias* field, we find $d_w = \alpha$. This is the first exactly soluble model displaying "anomalous ballistic diffusion," which we interpret physically in terms of a Lévy-flight-type random walk on a linear chain lattice.

How are the laws of electrical transport for a two-component random network changed when one of the conducting species has zero resistance? This question has recently been the object of many investigations,[1-8] focusing on the simple case in which the superconducting species is present in finite concentration p. A basic quantity which describes the transport properties of the system is the mean-square displacement $\langle x^2 \rangle$ of a random walker. The macroscopic diffusion constant $D = d\langle x^2 \rangle / dt$ is related to the electrical conductivity by the Einstein relation. In general, one finds $\langle x^2 \rangle \sim t^{2/d_w}$, with the diffusion exponent $d_w = 2$ in Euclidean lattices. Anomalous diffusion,[9-13] with d_w *greater than* 2, occurs when diffusion is slowed down in a self-similar fashion (e.g., if the diffusion is constrained to occur on a fractal substrate).[9] Also, random distributions of transition rates can increase the diffusion exponent (see, e.g., Refs. 8 and 14). In this article, we report a new physical system with a tunable "anomalous ballistic diffusion," for which d_w is *less than 2.*[14]

Our system is a $d = 1$ lattice, each bond of which is a superconducting element. The bonds of this lattice are randomly "diluted" by unit resistors, according to the rule that after every string of l superconducting bonds a unit-resistance bond is placed. The random variable l is chosen from the distribution

$$P(l) \sim l^{-(\alpha+1)}, \ (\alpha > 0) \ , \tag{1a}$$

where α is a tunable parameter. Thus, very small values of α correspond to extremely long strings of superconducting bonds. Notice that in this system the concentration of unit resistors is strictly zero, in contrast to the case of percolation, in which the unit-resistance bonds (empty bonds) appear with a nonzero concentration $1 - p \equiv \varepsilon$. In fact, the distance l between empty bonds for $d = 1$ percolation is distributed according to

$$P_0(l) \sim \exp(-l/\varepsilon) \ . \tag{1b}$$

Thus, the clusters of superconducting bonds are vastly larger in our system than in $d = 1$ percolation.

The superconductors in the mixture can be regarded as short circuits; thus, the resistance ρ scales as

$$\rho \sim N(L) \sim L^{\tilde{\zeta}} \ , \tag{2}$$

where $N(L)$ is the number of resistors included in a system of total length L and $\tilde{\zeta}$ is the resistivity exponent.

In order to find $N = N(L)$ we write

$$L = \sum_{i=1}^{N} l_i = N \int_0^{l_{\max}} l P(l) dl. \tag{3}$$

We must distinguish two regimes of α depending on the "range" of the probability distribution. For $\alpha \geq 1$ ("short range"), we recover the conventional result for a nonzero concentration of resistors, $L \sim N$. For $\alpha < 1$ ("long range"), however, the integral in (3) is dominated by its upper integration limit $l_{\max}$, which *itself* depends on N. To see this, we first note that if we choose N random numbers R, $0 \leq R \leq 1$, then

$$\min_{\{N\}} R = 1/N \ . \tag{4a}$$

Since R is homogeneously distributed, $P(l)dl \equiv 1 \times dR$ and we have

$$l^{-\alpha} = R \ . \tag{4b}$$

Combining (4a) and (4b), we find

$$l_{\max} \sim N^{1/\alpha} \ . \tag{5}$$

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Hence, (3) becomes $L \sim N(l_{\max})^{1-\alpha}$, so, in general,

$$L \sim \begin{cases} N, & \alpha \geq 1, \\ N^{1/\alpha}, & \alpha < 1. \end{cases} \tag{6}$$

Combining (2) and (6) we find

$$\tilde{\zeta} = \begin{cases} 1, & \alpha \geq 1, \\ \alpha, & \alpha < 1. \end{cases} \tag{7}$$

To study the diffusion exponent d_w, we note that in our model the walker performs a random walk on the resistors only (actually a type of Lévy flight[15]). Thus, the number N or resistors visited by the random walker scales as $N^2 \sim t$. Hence, from (6) we obtain $t \sim N^2 \sim L^{2\alpha}$ ($\alpha \leq 1$) and $t \sim N^2 \sim L^2$ ($\alpha > 1$). Therefore,

$$d_w = \begin{cases} 2, & \alpha \geq 1, \\ 2\alpha, & \alpha < 1. \end{cases} \tag{8}$$

Thus, for $\alpha < 1$ we obtain *ballistic* diffusion that is $d_w < 2$, while for $\alpha \geq 1$ we again obtain the conventional result $d_w = 2$ which is the value for a nonzero concentration of resistors.

When an external field E is applied on this system in any dimension, the time required to make a displacement L is proportional to the number of sites visited $N(L)$; i.e., $t \sim N(L) \sim L^{\alpha}$ for $\alpha \leq 1$, and $t \sim L$ for $\alpha > 1$. Hence,

$$d_w = \begin{cases} 1, & \alpha \geq 1, \\ \alpha, & \alpha < 1, \end{cases} \tag{9}$$

for biased diffusion in the superconducting-resistor mixture.

In summary, then, we have seen that in our model system we find critical exponents d_w and $\tilde{\zeta}$ that are (a) independent of a system parameter over a wide range of α values, sticking at their "classical" values $d_w = 2$ and $\tilde{\zeta} = 1$, for $\alpha > \alpha_c$, and (b) then become "unstuck" for $\alpha < \alpha_c$, varying continuously with α.

How general is this sort of phenomenon?

(i) A parallel phenomenon has been seen in the case of diffusion in a random resistor network where the resistance values are not all unity but rather are chosen from a power-law distribution $P(R) \sim R^{-(1+\alpha)}$, where $0 \leq R^{-1} \leq 1$; d_w was also found to stick at its classical value for $\alpha < \alpha_c$ and to become unstuck and vary continuously with α for $\alpha > \alpha_c$.[8,16,17]

(ii) Suppose one grows a percolation cluster by a method whereby sites are added to a cluster one at a time according to a power-law probability distribution which determines the growth sites chosen (the so-called "random butterfly" method). One finds that the fractal dimension d_g of the growth sites sticks at its classical value for $\alpha < \alpha_c$ and depends continuously on α for $\alpha > \alpha_c$.[18]

(iii) Consider diffusion in a random structure (like a random comb) where the dangling ends are distributed according to an exponential distribution.[13] Let the diffusion have a topological bias, with a probability $p_+ \propto 1 + E$ for taking a step that increases the path length from a source, and a probability $p_- \propto 1 - E$ for taking a step that decreases the path length. For $E < E_c$, d_w sticks at its classical value $d_w = 1$, while for $E > E_c$, d_w depends continuously upon E.[13]

We note, in conclusion, that, in general, *all* critical exponents depend on the *"system"* dimension d in a fashion quite analogous to the dependence upon α of the exponents for the systems considered here. For $d > d_c$, exponents stick at their mean-field values, while for $d > d_c$, they vary continuously with the tunable parameter d. Thus, our work suggests that one seek to identify the essential features that determine when a system has this characteristic behavior.

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[14]The existence of systems with anomalous ballistic diffusion (termed "superdiffusion" by some) was suggested recently [R. B. Pandey, J. Phys. A **17**, L551 (1984); J. Heinrichs and N. Kumar, J. Phys. C **17**, L741 (1984); J. Heinrichs, Phys. Rev. Lett. **52**, 1261 (1984)]. Ballistic transport was actually observed experimentally very recently in GaAs [M. Haiblum, M. I. Nathan, D. C. Thomas, and C. M. Knoedler, Phys. Rev. Lett. **55**, 2200 (1985)]. Also anomalously fast diffusion, with $d_w = \frac{3}{2} < 2$, was suggested for turbulent flows [S. Grossman, F. Webner, and K. H. Hoffman, J. Phys. (Paris) Lett. **46**, L575 (1985)].

[15]The difference between our model and the traditional Lévy-flight model is that in the Lévy flight *each* jump is taken from the distribution (1a). However, in our model if the random walker again visits the same site (resistor), the size of the jump will be the same as in the previous visits. It is quite possible that the Lévy flight, which has no memory, corresponds to the mean-field limit for our model, and that both models coincide above the critical dimension d_c ($d_c = 2$).

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