



On the size distribution of business firms

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Abstract

The size distribution of business firms is explained using number and size of firms' constituent components. It is a lognormal distribution multiplied by a stretching factor which can lead to a Pareto upper tail. This result is confirmed empirically.

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1. Introduction

Firm size distribution is the outcome of underlying firm growth dynamics involving entry and exit of firms as well as of their constituent units such as products. However, only a few scholars have been able to account for the size and the growth rate distribution of business firms simultaneously.¹ The difficulty derives from a puzzle caused by two stylized facts.

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¹ See Sutton (1997) and De Wit (2005) for reviews.

The first empirical regularity is that the size distribution satisfies a Pareto Law, at least in the upper tail. As early as in 1896, Pareto observed this systematic pattern in the distribution of personal incomes. In the subsequent works, the Pareto Law has been found to be consistent over time, across countries and size distributions of economic entities as diverse as individuals, cities and firms.

The second stylized fact dates back to [Gibrat \(1931\)](#) who stated that the firm growth rate does not depend upon the size of the firm: the so-called “Law of Proportionate Effect”. This fact has been confirmed in numerous empirical works ([Sutton, 1997](#)).

Unfortunately, the second fact cannot be easily reconciled with the first one since it is well known that in the limit, the proportional growth process gives rise to the lognormal distribution, not the Pareto distribution. However, there is a tradition in the economics of income inequality ([Champernowne, 1953](#)), industrial organization ([Ijiri and Simon, 1977](#); [Sutton, 1997](#)) and size distribution of cities ([Gabaix, 1999](#); [Eeckhout, 2004](#)) of models of proportionate growth that generate Pareto upper tails by assuming some version of a reflective lower bound.

The purpose of this paper is twofold. First, a new resolution of the puzzle is presented, which does not impose a lower bound on firm sizes. Moreover, our results hold at any level of aggregation of the economy from products to GDP (see [Fu et al., 2005](#)). Our model provides an accurate description of firm growth and size distributions based on the growth and size of the firms’ constituent components. Second, we reveal how it could be possible that the Pareto Law is repeatedly confirmed in the literature, while the underlying distribution is in fact close to a lognormal. The predictions of our model are confirmed empirically both at the level of products and companies.

2. The growth and size of business firms

As in [Fu et al. \(2005\)](#), we model business firms as economic entities consisting of a random number of units.² Two key sets of assumptions in the model are:

- the number of units in a firm grows in proportion to the existing number of units and
- the size of each unit grows in proportion to its size.

More specifically, the first set of assumptions is:

- (1) Each class α consists of $K_\alpha(t)$ units. At time $t=0$ there are $N(0)$ classes consisting of a total of $n(0)$ units. At each time step a new unit is created. Thus the number of units at time t is $n(t)=n(0)+t$.
- (2) With birth probability b , this new unit is assigned to a new class. With probability $1-b$, a new unit is assigned to an existing class α with probability $P_\alpha=(1-b)K_\alpha(t)/n(t)$.

The second set of assumptions of the model is:

- (3) At time t , each class α has $K_\alpha(t)$ units of size $\xi_i(t)$, $i=1, 2, \dots, K_\alpha(t)$ where K_α and $\xi_i > 0$ are independent random variables.

² See also [Ijiri and Simon \(1977\)](#), [Sutton \(1997\)](#), [De Fabritiis et al. \(2003\)](#).

(4) At time $t+1$, the size of each unit is decreased or increased by a random factor $\eta_i(t) > 0$ so that

$$\xi_i(t+1) = \xi_i(t)\eta_i(t), \tag{1}$$

where $\eta_i(t)$, the growth rate of unit i , is a random variable that is independent of all other η_i and ξ_i values.

Basing on the first set of assumptions, we find the probability distribution of the number of units in the classes $P(K)$. Then, we make use of the second set of assumptions to derive the size distribution of business firms $P(S)$ given $P(K)$. In [Fu et al. \(2005\)](#) and [Yamasaki et al. \(2006\)](#) we have characterized the distribution of the number of the constituent units (K) and growth rates (g) in a version of the preferential attachment model originally proposed by [Ijiri and Simon \(1977\)](#). There, we have described the distribution $P(K)$ in the entire range of K and found it to be a power law with an exponential cut-off. We have shown that the exponential cut-off of the power law is not due to finite size but is an effect of the finite time interval of the evolution. Herein, we shall use those results when deriving the long-run distribution of the size S of the economic entities (see also [De Fabritiis et al., 2003](#)). Let us first calculate the long-run size distribution of firms which sell a given number of products K , i.e., let $t \rightarrow \infty$ holding K fixed. Three distinct cases emerge.

Case $K=1$. In this case, the Gibrat Law works perfectly. Since we know that $\ln \xi_{t+1} = \ln \xi_t + \ln \eta_t = \sum_{\tau=0}^t \ln \eta_\tau$, from the Central Limit Theorem we obtain

$$\frac{\sum_{\tau=0}^t \ln \eta_\tau - t\mu_\eta}{\sqrt{tV_\eta}} \xrightarrow{t \rightarrow \infty} N(0,1). \tag{2}$$

This means that the distribution of ξ_t approaches the lognormal limit – for sufficiently large t , approximately $\ln \xi_t \sim N(t\mu_\eta, tV_\eta)$. For large t , we also find that $\mu_\xi = E\xi_t = e^{t(m_\eta + V_\eta/2)}$, and $\sigma^2 = \text{Var}(\xi_t) = e^{2t(m_\eta + V_\eta/2)}(e^{tV_\eta} - 1)$.

Case $K > 1$. In that case, $S_t = \sum_{i=1}^K \xi_t^i$ is a sum of random variables that are asymptotically lognormally distributed. A sum of lognormals does not have a closed form. Bearing in mind that any approximation to a sum of lognormals which uses the lognormal itself is erroneous in the body of the distribution, we use [Slimane \(2001\)](#) bounds as our first approximation.

By definition, we have that $\mathcal{P}(S|K) \equiv \mathcal{P}(\sum_{i=1}^K \xi_t > S) = 1 - F_{\xi}^{\otimes K}(S)$, where $\otimes K$ denotes the K -fold convolution of the F distribution of ξ with itself. Slimane’s bounds tell us that for large t ,

$$1 - \left[\Phi \left(\frac{\ln S - m_S}{\sqrt{V_S}} \right) \right]^K \leq \mathcal{P}(S|K) \leq 1 - \left[\Phi \left(\frac{\ln (S/K) - m_S}{\sqrt{V_S}} \right) \right]^K, \tag{3}$$

where Φ denotes the CDF of the standardized Gaussian distribution, and where $m_S = E(\ln \xi_t)$; $V_S = \text{Var}(\ln \xi_t) = tV_\eta$.

Case $K \rightarrow \infty$. From the Central Limit Theorem we obtain that

$$\frac{\sum_{i=1}^K \xi_t - K\mu_\xi}{\sqrt{K}\sigma_\xi} \xrightarrow{K \rightarrow \infty} N(0,1). \tag{4}$$

As the number of products per firm goes to infinity, the size distribution of firms approaches a Gaussian distribution with mean $\mu_S = K\mu_\xi = Ke^{(m_\eta + V_\eta/2)}$ and variance $\sigma_S^2 = K\sigma_\xi^2 = Ke^{2t(m_\eta + V_\eta/2)}(e^{V_\eta} - 1)$. We find that μ_S and σ_S increase linearly with K but exponentially with t . This means that the convergence to the lognormal, due to the Gibrat process, is much easier and faster than the convergence to the normal due to an ever increasing number of marketed products per firm.

Let us now proceed to the calculation of the actual long-run firm size distribution, which allows K to grow over time, indicating the net increase in the total number of products.

In the case where $b=0$ and there is no product entry, the only force at play is the Gibrat process. As size of each product ξ^i approaches a lognormal distribution, the size of each firm approaches a distribution which is a sum of K lognormals, where K is a predetermined constant. Initial conditions in terms of number of products per firm play a crucial role here. If, however, new products arrive, then all firms have in the limit an infinite number of products. $K \rightarrow \infty$ for all firms, and thus S_t approaches a Gaussian distribution with mean μ_S and variance σ_S^2 .

The most interesting case emerges if $b > 0$. To get the long-run firm size distribution, one has then to find

$$\mathcal{P}(S) = \sum_{K=1}^{\infty} P(K)\mathcal{P}(S|K). \tag{5}$$

Asymptotically, all units (firms) are new. Approximating $\mathcal{P}(S|K)$ by a mixture of Slimane’s upper and lower bounds, letting $b \rightarrow 0$ and $t \rightarrow \infty$, and using $P(K) = \frac{1}{K^2} (1 - (K + 1)e^{-K})$ yields the following CDF:

$$\mathcal{P}(S) \approx \sum_{K=1}^{\infty} \frac{1}{K^2} (1 - (K + 1)e^{-K}) \left\{ 1 - \left[\Phi \left(\frac{\ln(S/K^\gamma) - m_S}{\sqrt{V_S}} \right) \right]^K \right\}, \tag{6}$$

with $\gamma \in [0, 1]$.

Denoting $\Phi \left(\frac{\ln(S/K^\gamma) - m_S}{\sqrt{V_S}} \right) \equiv h(S)$ for simplicity, and differentiating Eq. (6) with respect to S yields the following complementary CDF $P(S)$

$$P(S) = -\mathcal{P}'(S) = \underbrace{h'(S)}_{\text{lognormal}} \times \underbrace{\sum_{K=1}^{\infty} \frac{h(S)^{K-1}}{K} (1 - (K + 1)e^{-K})}_{\text{stretching factor}}. \tag{7}$$

The firm size PDF is a PDF of a lognormal distribution multiplied by a stretching factor which increases with S . For very small S , the stretching factor becomes negligible and the distribution is close to a lognormal. The larger the S value, driving $h(S)$ towards unity, the more powerful the stretching factor.

Replacing summation with integration, we obtain that the stretching factor can be approximated by $\sum_{K=1}^{\infty} \frac{h(S)^{K-1}}{K} (1 - (K + 1)e^{-K}) \approx \int_1^{\infty} \frac{h(S)^{x-1}}{x} (1 - (x + 1)e^{-x}) dx \approx \int_1^{\infty} \frac{h(S)^{x-1}}{x} dx \approx \int_1^{\infty} \frac{h(S)^{x-1}}{x} dx \approx \int_0^{\infty} h(S)^x dx = -\frac{1}{\ln h(S)}$. See that for large S , with $h(S) \rightarrow 1$, the stretching factor can become arbitrarily large.

Fig. 1 depicts size distributions for products and companies in the worldwide pharmaceutical industry. It is based on a unique database which records sales figures of 340,560 products commercialized by 8072 firms in 28 countries from 1994 to 2004, covering the whole size distribution for products and firms and monitoring the flows of entry and exit at both levels. At a first glance, both distributions depict a lognormal shape but at a closer look the firm size distribution reveals a power law departure in the upper tail. Numerical analysis confirms that indeed the right tail is asymptotically a power law. Both distributions are well fitted by Eq. (7) with $\gamma=0$ and $\gamma=1/2$ respectively.

3. Discussion

The size of a firm is equal to the sum of the size of its constituent components. Fig. 1 reveals that the size of products is lognormally distributed while the size distribution of business firms shows a departure in the upper tail which decays as a power law. Both the “Law of Proportionate Effect” and the Pareto Law hold since the Central Limit Theorem does not work effectively in case of lognormally distributed random variables. The sum of lognormally distributed random variables does not have a closed form, and several possible approximations have been proposed for the first two moments, which involve series evaluations. These estimates are all based on the approximation that a sum of lognormals is still a lognormal

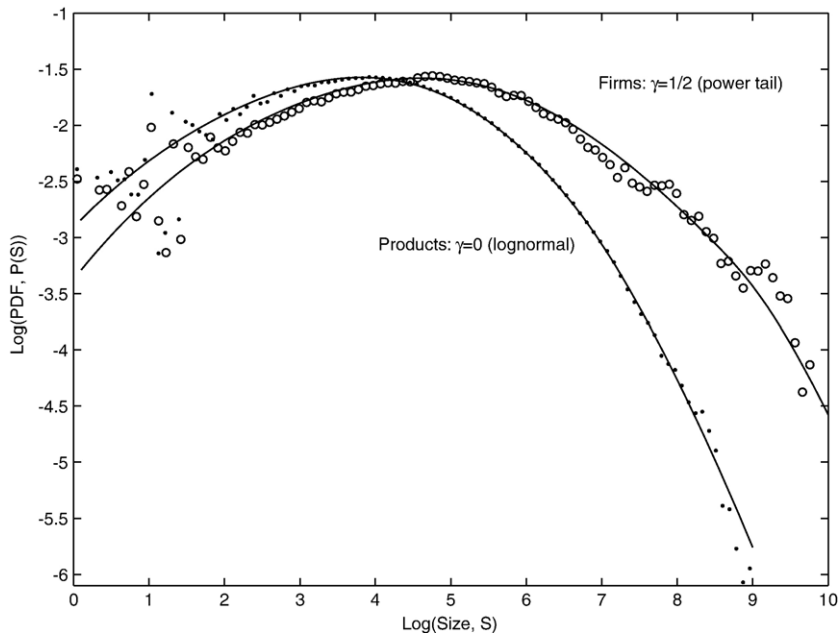


Fig. 1. Product (●) and firm (○) size distributions fitted by Eq. (7) with $\gamma=0$ (pure lognormal) and $\gamma=1/2$ (lognormal with a Pareto tail) respectively. For the distribution of product sizes, we replace the upper limit of summation in Eq. (7) with $K_{\max} = 1$; for firm sizes, we use $K_{\max} = 1348$ which is the maximum product range found in our dataset. We also use the ML estimates $m_S = 8.5141$ and $V_S = 3.2131$ of the parameters of the lognormal distribution of products for both cases. We note that a lognormal fit ($\gamma=0$ with $K_{\max} = 1348$) fails to account for the heavy tail of the firm size distribution.

distribution, stable upon aggregation. In fact, a lognormal distribution $P(S)$ with parameters μ and σ behaves as a power law between S^{-1} and S^{-2} for a wide range of its support $S_0 < S < S_0 e^2 \sigma^2$, where S_0 is a characteristic scale corresponding to the median (Sornette, 2000). Since a decay similar to a power law is present for a large part of the upper tail, the Central Limit Theorem does not work properly (De Fabritiis et al., 2003). This argument explains the apparent stability of the size distribution upon aggregation. In particular, simple numerical simulations show that $\log(\sum_j S_j)$ depicts a Pareto $1/S$ tail, in line with the empirical distributions.

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