Robustness of a network formed by n interdependent networks with a one-to-one correspondence of dependent nodes

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Many real-world networks interact with and depend upon other networks. We develop an analytical framework for studying a network formed by n fully interdependent randomly connected networks, each composed of the same number of nodes N. The dependency links connecting nodes from different networks establish a unique one-to-one correspondence between the nodes of one network and the nodes of the other network. We study the dynamics of the cascades of failures in such a network of networks (NON) caused by a random initial attack on one of the networks, after which a fraction p of its nodes survives. We find for the fully interdependent loopless NON that the final state of the NON does not depend on the dynamics of the cascades but is determined by a uniquely defined mutual giant component of the NON, which generalizes both the giant component of regular percolation of a single network ($n = 1$) and the recently studied case of the mutual giant component of two interdependent networks ($n = 2$). We also find that the mutual giant component does not depend on the topology of the NON and express it in terms of generating functions of the degree distributions of the network. Our results show that, for any $n \geqslant 2$ there exists a critical $p = p_c > 0$ below which the mutual giant component abruptly collapses from a finite nonzero value for $p \geqslant p_c$ to zero for $p < p_c$, as in a first-order phase transition. This behavior holds even for scale-free networks where $p_c = 0$ for $n = 1$. We show that, if at least one of the networks in the NON has isolated or singly connected nodes, the NON completely disintegrates for sufficiently large n even if $p = 1$. In contrast, in the absence of such nodes, the NON survives for any n for sufficiently large p. We illustrate this behavior by comparing two exactly solvable examples of NONs composed of Erdős-Rényi (ER) and random regular (RR) networks. We find that the robustness of n coupled RR networks of degree k is dramatically higher compared to the n-coupled ER networks of the same average degree $\bar{k} = k$. While for ER NONs there exists a critical minimum average degree $\bar{k} = \bar{k}_{\min} \sim \ln n$ below which the system collapses, for RR NONs $k_{\min} = 2$ for any n (i.e., for any $k > 2$, a RR NON is stable for any n with $p_c < 1$). This results arises from the critical role played by singly connected nodes which exist in an ER NON and enhance the cascading failures, but do not exist in a RR NON.

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I. INTRODUCTION

Infrastructures, which affect all areas of modern life, are usually interdependent. Examples include electric power, natural gas and petroleum production and distribution, telecommunications, transportation, water supply, banking and finance, emergency and government services, agriculture, and other fundamental systems and services which are critical to security and economic prosperity. Recent disasters ranging from hurricanes to large-scale power blackout and terrorist attacks have shown that significant dangerous vulnerability arises from the many interdependencies across different infrastructures [1–5]. Infrastructures are frequently connected at multiple points through a wide variety of mechanisms, such that a bidirectional relationship exists between the states of any given pair of connected networks. For example, in California, electric power disruptions in early 2001 affected oil and natural gas production, refinery operations, pipeline transport of gasoline and jet fuel within California and its neighboring states, and the movement of water from northern to central and southern regions of the state for crop irrigation. Another dramatic real-world example of a cascade of failures is the electrical blackout that affected much of Italy on 28 September 2003: the shutdown of power stations directly led to the failure of nodes in the Supervisory Control and Data Acquisition (SCADA) communication network, which in turn caused further breakdown of power stations [5,6]. Identifying, understanding, and analyzing such interdependencies are therefore significant challenges. These challenges are greatly magnified by the breadth and complexity of our modern critical national interdependent infrastructures [4].

In recent years we have witnessed important advances in the field of complex networks [7–19]. The internet, airline routes, and electric power grids are all examples of networks whose function relies crucially on the connectivity between the network components. An important property of such systems is their robustness to node failures. Almost all research has been concentrated on the case of a single or isolated network which does not interact with or depend on other networks. Recently, based on the motivation that modern infrastructures are becoming significantly more dependent on each other, a system of two interdependent networks has been studied [6,20–22]. A fundamental property of interdependent networks is that the failure of a node in one network may lead to the failure of dependent nodes in other interdependent networks, which in turn may cause further damage in the first network and so on, leading to a global cascade of failures. Reference [6] developed a framework for analyzing the robustness of two interacting networks subject to such cascading failures. They found that interdependent networks behave very differently from single networks and become significantly more vulnerable compared to their noninteracting counterparts. The case of a partially interdependent pair of networks was studied [20].

More recently, two important generalizations of the basic model [6] have been developed. (i) Because in real-world scenarios the initial failure of important nodes ("hubs") may not be random but targeted, a mathematical framework for understanding the robustness of interdependent networks under an initial targeted attack has been studied [23]. They developed a general technique that uses the random-attack problem to map the targeted-attack problem in interdependent networks. (ii) Also in real-world scenarios, the assumption that each node in network A depends on one and only one node in network B and vice versa may not be valid. To correct this shortcoming, a theoretical framework for understanding the robustness of interdependent networks with a random number of support and dependence relationships has been developed and studied [24].

In all of the above studies [6,20,23,24], the dependent pairs of nodes in both networks were assumed to be chosen randomly. Thus, when high degree nodes in one network depend with a high probability on low degree nodes of another network, the configuration becomes vulnerable. To quantify and better understand this phenomenon, Ref. [25] proposed two "intersimilarity measures" between the coupled nodes. Intersimilarity occurs in coupled networks when (a) nodes with similar degrees tend to be interdependent and (b) the neighboring nodes of interdependent nodes in each network also tend to be dependent. They found that, as the coupled networks become more intersimilar, the system becomes more robust [25,26]. Reference [25] also studies a system composed of the interdependent world-wide seaport network and the world-wide airport network. They found indeed that well-connected seaports tend to couple with well-connected airports. The case in which all pairs of interdependent nodes in both networks have the same degree was solved analytically in Ref. [27].

In many realistic examples, more than two networks depend on each other. For example, diverse infrastructures such as water and food supply, communications, fuel, financial transactions, and power stations are coupled together [2,3,5,28]. Understanding the vulnerability due to such interdependencies is a major challenge for designing resilient infrastructures.

We study here a model system [29,30] comprising a network formed by n fully interdependent networks, where each network consists of N nodes (see Fig. 1). Each of the N nodes in one network is connected to a node in another network by bidirectional dependency links, thereby establishing a one-to-one correspondence. We develop a mathematical framework [29] to study the robustness of tree-like "network of networks" (NON) by studying the dynamical process of the cascading failures. We find an exact analytical law for percolation of a NON system composed of n coupled randomly connected networks. Our result generalizes the known Erdős-Rényi (ER) [31–33] result as well as the random regular (RR) result for the giant component of a single network. We find that, while for $n = 1$ the percolation transition is second order, for $n > 1$ cascading failures occur and the transition becomes a first-order transition. Our results for n interdependent networks show that the classical percolation theory extensively studied in physics and mathematics is in fact a limiting case of the richer, more general, and very different percolation law which holds for realistic interacting networks.

Additionally, we show for both ER and RR NONs that

(i) for any loopless topology of NON, the critical percolation threshold and the giant component depend only on the *number* of networks involved and their degree distributions but *not on the interlinked topology* (Fig. 1),

(ii) the robustness of NONs *significantly decreases* with n, and

(iii) for a network of n ER networks, all with the same average degree k, there exists a minimum degree $k_{\min}(n)$ increasing with n, below which $p_c = 1$ (i.e., for $k < k_{\min}$ the NON will collapse once any finite number of nodes fail).

The analytical expression for $k_{\min}(n)$ generalizes the known result $k_{\min}(1) = 1$ for ER below which the network collapses. In sharp contrast, a NON composed of RR networks is significantly more robust. In the RR NON case there is no $k_{\min}$ which is independent of n below which the NON collapses ($k_{\min} = 2$ for all n). This is due to the multiple links of each node in the RR system compared to the existence of singly connected nodes in the ER case. We also discuss (Sec. VII) the critical effect of singly connected nodes on the vulnerability of the NON ER structure.

II. GENERATING FUNCTIONS FOR SINGLE NETWORK

We begin by describing the generating function formalism for a single network that will be useful in studying interdependent networks. We assume that all N_i nodes in network i are randomly assigned a degree k from a probability distribution $P_i(k)$ and are randomly connected with the only constraint that the node with degree k has exactly k links [34]. We define the generating function of the degree distribution

$$G_i(x) = \sum_{k=0}^{\infty} P_i(k)x^k, \tag{1}$$

where x is an arbitrary complex variable. The average degree of network i is

$$\bar{k}_i \equiv \sum kP_i(k) = G_i'(1). \tag{2}$$

In the limit of infinitely large networks, $N_i \to \infty$, the random connection process can be modeled as a branching process in which an outgoing link of any node has a probability $kP_i(k)/\bar{k}_i$ to be connected to a node with degree k, which in turn has $k - 1$ outgoing links. The generating function of this branching process is defined as

$$H_i(x) \equiv \frac{\sum_{k=0}^{\infty} P_i(k)kx^{k-1}}{\bar{k}_i} = \frac{G_i'(x)}{G_i'(1)}. \tag{3}$$

The probability f_i that a randomly chosen outgoing link does not lead to an infinitely large giant component satisfies a recursion relation $f_i \equiv H_i(f_i)$. Accordingly, the probability that a randomly chosen node does belong to a giant component is given by $g_i \equiv 1 - G_i(f_i)$. Once a fraction $1 - p$ of nodes is randomly removed from a network, its generating function remains the same but must be computed from a new argument $z = pf_i + 1 - p$ [35–37]. Thus $P_{\infty,i}$, the fraction of nodes

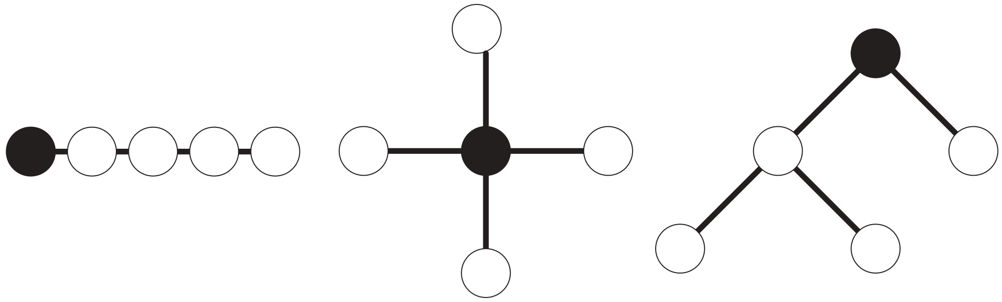

FIG. 1. Three types of loopless NONs, each composed of five coupled networks. They all have the same percolation threshold and the same giant percolation component.

which belongs to the giant component, is given by [36–38]

$$P_{\infty,i} = p g_i(p), \tag{4}$$

where

$$g_i(p) = 1 - G_i(p f_i(p) + 1 - p), \tag{5}$$

and $f_i(p)$ satisfies

$$f_i(p) = H_i(p f_i(p) + 1 - p). \tag{6}$$

As p decreases, the nontrivial solution $f_i < 1$ of Eq. (6) gradually approaches the trivial solution $f_i = 1$. Accordingly, $P_{\infty,i}$ gradually approaches zero as in a second-order phase transition and becomes zero when the two solutions of Eq. (6) coincide at $p = p_c$. At this point the straight line corresponding to the left-hand side of Eq. (6) becomes tangential to the curve corresponding to its right-hand side, yielding

$$p_c = \frac{1}{H'_i(1)}. \tag{7}$$

For example, for Erdős-Rényi (ER) networks [31–33], characterized by a Poisson degree distribution,

$$G_i(x) = H_i(x) = \exp[\bar{k}_i(x-1)], \tag{8}$$

$$g_i(p) = 1 - f_i(p), \tag{9}$$

$$f_i(p) = \exp\{p\bar{k}_i[f_i(p) - 1]\}, \tag{10}$$

and

$$p_c = \frac{1}{\bar{k}_i}. \tag{11}$$

Finally, using Eqs. (4), (9), and (10) we can obtain a direct equation for $P_{\infty,i}$:

$$P_{\infty,i} = p[1 - \exp(-\bar{k}_i P_{\infty,i})]. \tag{12}$$

III. DYNAMIC PROCESS OF CASCADING FAILURES

In this paper we will study a particular variant [6] of the dependency among the networks participating in the NON, namely, bidirectional dependency links establishing one-to-one correspondence between the nodes of all networks in the NON. Other variants, which include autonomous nodes [20] and multiple dependency links [24], can be studied along the same lines. We assume that the NON consists of n networks each having N nodes (Fig. 1). Each node in Fig. 1 represents a network, and each link between two networks i and j denotes the existence of a one-to-one dependency between all the nodes of the linked networks. The functioning of one node in network i depends on the functioning of one and only one node in network j ($i, j \in \{1,2,\ldots,n\}$, $i \neq j$), and *vice versa* (bidirectional links). If node i in network A stops functioning, the dependent node j in network B stops functioning within the time of autonomous functioning, τ_a.

If the NON has a tree-like topology, the dependency links establishing one-to-one correspondence between the pairs of directly linked networks establish a unique one-to-one correspondence between the nodes of any two networks of the NON. The removal of a single node in one network causes the removal of the set of all n correspondent nodes each belonging to a different network. If we assume the existence of such a unique one-to-one correspondence, our treatment applies not only to a tree-like NON but to a NON of any topology.

On the other hand, if in the NON there is a mismatch in the correspondence of the nodes in the dependency links forming a loop, a failure of a single node may cause a complete collapse of all the networks forming a loop. Indeed if any node A_i stops functioning then a different node A_{t_i} in the same network will stop functioning. If t_i is a permutation of the nodes $i = 1,2,3\ldots N$, then all the nodes forming a cycle in this permutation which includes node i will stop functioning. The probability for a randomly selected element to belong to a cycle of length ℓ in a random permutation of N elements follows a uniform distribution $P(\ell) = 1/N$. As $N \to \infty$, the mathematical expectation of a fraction of elements that do not belong to the cycles to which $k = pN$ randomly selected elements belong scales as

$$\frac{1}{k+1} \sim \frac{1}{pN} \to 0. \tag{13}$$

Thus, removal of an infinitesimally small fraction of nodes from a NON with a loop completely eliminates all the networks in a loop if no assumptions on the nature of the permutation created by the mismatch are made. Once a single network in the NON will stop functioning, all other networks in the NON will stop functioning and hence the NON with a loop will completely disintegrate unless there is a unique one-to-one correspondence established by the dependency links among the nodes of each network, or the permutation characterizing the mismatch in the dependency links along the loop is not random. Here, we restrict ourselves to the simple case of the one-to-one correspondence.

We assume that, within network i, the nodes are randomly connected by connectivity links with degree distribution $P_i(k)$. We further assume that only the nodes belonging to the giant connected cluster of each network can function. Other nodes which belong to smaller clusters become nonfunctional within a time τ_c. For simplicity, we assume that τ_c are equal for all nodes in all networks and that $\tau_a \ll \tau_c$. However, the final state of the model does not depend on these details, since it is completely defined by the mutually connected clusters (i.e., the clusters of the correspondent nodes in each network, which are independently connected by the connectivity links of each network of the NON).

Once a fraction of nodes $1 - p$ is removed from a single network, which we will call the root, the corresponding dependent nodes in all networks become nonfunctional within a short time interval bounded by

$$t_0 \equiv n\tau_a \ll \tau_c. \tag{14}$$

At time $t_1 = \tau_c + t_0 \approx \tau_c$, the nodes which do not belong to the giant components of the individual networks stop functioning. Since the dependency links are random, the fraction of corresponding nodes which simultaneously belongs to the giant components in all networks is

$$\mu(t_1) = p \prod g_i[x_i(t_0)], \tag{15}$$

where $x_i(t_0) = p$ for every network and the functions $g_i(x_i)$ are defined by Eq. (9) in which $p = x_i$. From the point of view of each individual network this is equivalent to a random attack after which the fraction

$$x_i(t_1) = \frac{\mu(t_1)}{g_i[x_i(t_0)]} \tag{16}$$

randomly survives from the entire network, since the fraction $\mu(t_1)$ of survived nodes is selected from the current giant component with fraction $g_i[x_i(t_0)]$. Accordingly, at time $t_2 = t_1 + \tau_c$ only the nodes of the new giant component of each network $g_i[x_i(t_1)]$ remain functional and only the set of corresponding nodes which simultaneously belong to the giant components of each network will remain functional after time τ_a. The fraction of these nodes is

$$\mu(t_2) = p \prod g_i[x_i(t_1)]. \tag{17}$$

Thus at each stage of the cascade, the fraction of nodes that remains functional is

$$\mu(t_{n+1}) = p \prod g_i[x_i(t_n)], \tag{18}$$

where $t_{n+1} = t_n + \tau_c$ and

$$x_i(t_n) = \frac{\mu(t_n)}{g_i[x_i(t_{n-1})]}. \tag{19}$$

At the end of the cascade, no further failures occur and $\mu_\infty = \mu(t_{n+1}) = \mu(t_n)$ is the mutual giant component of the NON. Obviously, $x_i(t_n) = x_i(t_{n+1}) \equiv x_i$, so the final state of the NON obeys $n+1$ equations with $n+1$ unknowns $x_1, x_2, \ldots, x_n$, and μ_∞, where

$$x_i = \mu_\infty / g_i(x_i), \tag{20}$$

and

$$\mu_\infty = p \prod_i^n g_i(x_i). \tag{21}$$

A different cascade of failures leading to the same final state emerges if $\tau_a \gg \tau_c$. This cascade is easy to describe if τ_a are equal for all the nodes in all networks and if a NON has a tree-like topology in which the shortest path distance D_{ij} between any two networks of the NON can be uniquely defined. We initialize $x_i(t) = 1$ for $t \leqslant 0$. The root of the NON is the network $i = 1$ from which a fraction $1 - p$ of nodes are removed due to random failure. Accordingly, we set $x_1(0) = p$. Computing the subsequent failures at times $0, \tau_a, 2\tau_a, \ldots$ starting from the root, we can show that the number of nodes that remain functional in each network at time $t > 0$ is

$$\mu_i(t) = x_i(t) g_i[x_i(t)], \tag{22}$$

where

$$x_i(t) = \min\left(p \prod_{j=1, j\neq i}^{n} g_j[x_j(t - D_{ij}\tau_a)], x_i(t - \tau_a)\right). \tag{23}$$

In this cascade, the state of each network changes on every second stage of the cascade. For example the state of the root $(i = 1)$ changes at time

$$t = 0, 2\tau_a, 4\tau_a, \ldots, \tag{24}$$

while the state of the networks in the kth shell of the root with $D_{j1} = k$, changes at times

$$t = k\tau_a, (k+2)\tau_a, (k+4)\tau_a, \ldots. \tag{25}$$

For $t \to \infty$ Eqs. (22) and (23) become equivalent to Eqs. (21) and (20). Figure 2 shows the dynamic of cascading failures at different time stages, and Fig. 3 shows how damage spreads in a NON system.

Simulations of the cascading failures in the tree-like NON of different topologies shown in Fig. 1 for the case $\tau_c \ll \tau_a$ agree well with Eqs. (22) and (23). In Figs. 4 and 5 we compare our theoretical results, Eqs. (23) and (22), with simulation results for 3 different types of NON: ER networks, RR networks, and SF networks. We find that while the dynamics is different for the three topologies shown in Fig. 1, the final P_∞ is the same as predicted by the theoretical results, Eqs. (21) and (20).

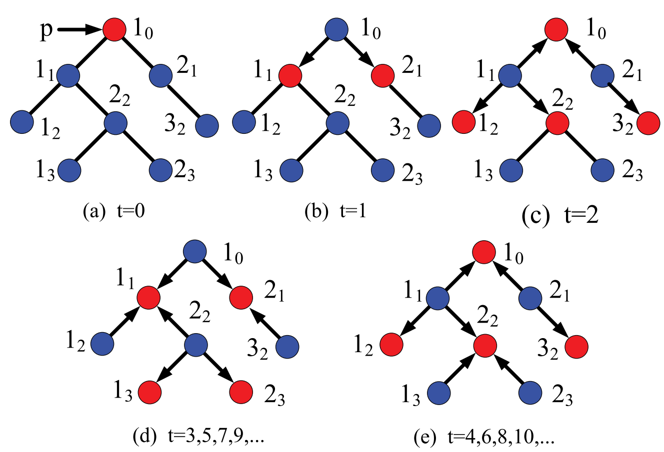

FIG. 2. (Color online) Dynamics of cascading failures at different time stages. In this figure, each node represents a network. The arrow (on the link) illustrates the direction of damage spreading from the root network to the whole NON shell by shell.

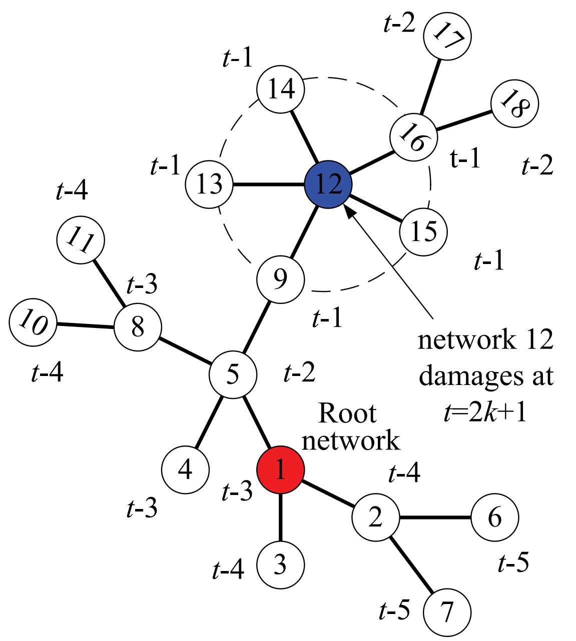

FIG. 3. (Color online) How does the damage spread in a NON system? In this figure, each node represents a network. When looking at network 12 for example, it becomes damaged at $t = 2k + 1$ ($k = 1, 2, 3, \ldots$). It receives the damage from network 8 at $t = 2k + 3$, because network 8 gets damage at $t = 2$ for the first time and its damage spreads to network 12 at $t = 5$ for the first time. This agrees with Eqs. (23) and (22) that network i receives damage from network j if and only if $t - D_{ij} \geqslant D_{1j}$.

Next we study the final steady state of the NON. Equations (20) and (21) can be simplified if we introduce a new variable [38]:

$$z_i = f_i x_i + 1 - x_i. \tag{26}$$

Using Eqs. (5) and (6) we obtain

$$f_i = H_i(z_i), \tag{27}$$

$$x_i = (1 - z_i)[1 - H(z_i)] \tag{28}$$

and

$$1 - z_i = p[1 - H_i(z_i)] \prod_{j=1, j\neq i}^{n} (1 - G_j(z_j)), \tag{29}$$

$$P_\infty \equiv \mu_\infty = p \prod_{i=1}^{n} (1 - G_i(z_i)) = \frac{[1 - G_i(z_i)](1 - z_i)}{1 - H_i(z_i)} \equiv F_i(z_i). \tag{30}$$

One can show that, if $\bar{k}_i$ exists, the functions $F_i(z_i)$ are analytical functions for $|z_i| < 1$ and are monotonically decreasing from $\bar{k}_i[1 - P_i(0)]/[\bar{k}_i - P_i(1)]$ at $z_i = 0$ to zero at $z_i = 1$ (see Sec. VII). Selecting i such that $F_i(0)$ has the smallest value, we can solve equations

$$F_j(z_j) = F_i(z_i) \tag{31}$$

with respect to z_j, and $z_j(z_i)$ can be substituted in Eq. (29) as

$$\frac{1}{p} = \frac{\prod_{j=1}^{n}[1 - G_j(z_j(z_i))]}{F(z_i)} \equiv R_i(z_i). \tag{32}$$

The right-hand side of Eq. (32), $R_i(z_i)$, is an analytical function for $z_i \in [0,1]$, such that $R_i(0) \leqslant 1$ and $R_p(z_i) \to 0$ for $z_i \to 1$. If its maximal value R_c in this interval is greater than unity, Eq. (32) has roots for $z \in [0,1)$ for $1/R_c \leqslant p \leqslant 1$. The smallest of these roots gives the physically meaningful solution from which the mutual giant component $1 > P_\infty > 0$ can be found from Eq. (30). The minimal $p = p_c = 1/R_c$ below which the solutions cease to exist corresponds to the maximum of $R_i(z_i)$. The point z_i^c at which this maximum is achieved satisfies the equation

$$\frac{dR_i(z_i^c)}{dz_i^c} = 0. \tag{33}$$

It can be shown (see Sec. VII), that if at least for one network $P_i(0) + P_i(1) > 0$ (condition I), and there exist two constants $M > 0$ and $\eta > 0$ such that for each network $\sum_{k<M} P_i(k) > \eta$ (condition II), then for sufficiently large n, $R_i(z_i)$ is less than unity for any $z \in [0,1]$ and hence a physically meaningful solution of Eq. (32) does not exist. In other words, if at least for one network there exist isolated or singly connected nodes, the NON completely disintegrates even for fully intact networks for sufficiently large n. This happens because a network which has a finite fraction of isolated and singly connected nodes by necessity has a finite fraction of nodes which do not belong to its giant component for $p = 1$ either because these nodes do not have links at all, or because they form the pairs of singly connected nodes linked to each other. These nodes will become nonfunctional at the first stage of the cascade and will cause the death of a finite fraction of nodes in each network due to-one-one correspondence of the interdependent nodes in all networks. Because we assume that the networks are randomly connected, these dead nodes will cause a disconnection of a finite fraction of nodes in all the networks. Again due to one-to-one correspondence of interdependent nodes, the death of these disconnected nodes will cause the death of a union of n finite sets of dependent nodes in any of the networks. Since from the point of view of each network, these n sets are randomly selected, for sufficiently large n, the fraction of nodes that does not belong to this union will become less than the percolation threshold for a certain network. Hence, this network will entirely collapse and, by necessity, will cause the collapse of the entire NON.

In contrast, it can be shown that if for all networks $P_i(0) + P_i(1) = 0$, then for any n, there exists $p(n) < 1$ such that for $1 \geqslant p > p(n)$ the mutual giant component of the NON exists (see Sec. VII). In other words, if there are no singly connected or isolated nodes in the entire NON, the mutual giant component will survive for sufficiently large p for any n. A simple physical reason for this is that, in the absence of isolated and singly connected nodes, small isolated clusters must contain finite loops, the chances of which in a randomly connected network are infinitesimally small for $N \to \infty$. Therefore, for $p = 1$, the giant component of each network coincides with the entire network, and hence the cascade that destroys the NON can start only for $p < 1$. It can be also proven that under condition II, $p(n) \to 1$ for $n \to \infty$.

For the case of n coupled networks when all networks are with the same degree distribution, all $G_i = G_0$ and $H_i = H_0$

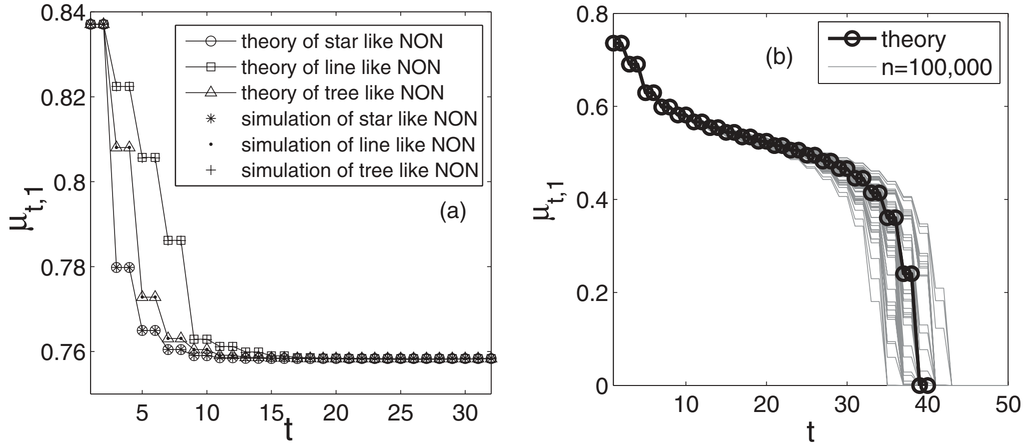

FIG. 4. (a) Simulation results of giant component of root network $\mu_{t,1}$ after t cascading failures for three types of NON composed of the 5 ER networks shown in Fig. 2. For each network in the NON, $N = 100\,000$ and $\bar{k} = 5$. The chosen value of p is $p = 0.85$, and the predicted threshold is $p_c = 0.764\,49$ [from Eqs. (44) and (47)]. All points are the results of averaging over 40 realizations. Note that while the dynamics is different for the three topologies, the final $P_\infty \equiv \mu_{\infty,1}$ is the same (i.e., the final P_∞ does not depend on the topology of the NON). (b) Simulations of the giant component, $\mu_{t,1}$, for the tree-like NON (Fig. 2) with the same parameters as in (a) but for $p = 0.755 < p_c = 0.764\,49$. The figure shows 50 simulated realizations of the giant component left after t stages of the cascading failures compared with the theoretical prediction of Eqs. (8), (22), and (23).

and Eqs. (30)–(33) can be simplified as

$$\frac{1}{p} = \frac{[1 - G_0(z_0)]^{n-1}[1 - H_0(z_0)]}{1 - z}, \tag{34}$$

$$P_\infty = p[1 - G_0(z_0)]^n = \frac{[1 - G_0(z_0)](1 - z_0)}{1 - H_0(z_0)}, \tag{35}$$

and

$$1 = \frac{\left(1 - z_0^c\right) H_0'\left(z_0^c\right)}{1 - H_0\left(z_0^c\right)} + \left(1 - z_0^c\right) \frac{(n-1) k H_0\left(z_0^c\right)}{1 - G_0\left(z_0^c\right)}. \tag{36}$$

Thus we obtain the critical threshold p_c as

$$p_c = \frac{1 - z_0^c}{\left[1 - G_0\left(z_0^c\right)\right]^{n-1}\left[1 - H_0\left(z_0^c\right)\right]}. \tag{37}$$

IV. CASE OF NETWORK OF NETWORKS COMPOSED OF N ER NETWORKS

A. General case

The case of a NON composed of n Erdős-Rényi (ER) [31–33] networks with average degrees $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_i, \ldots, \bar{k}_n$

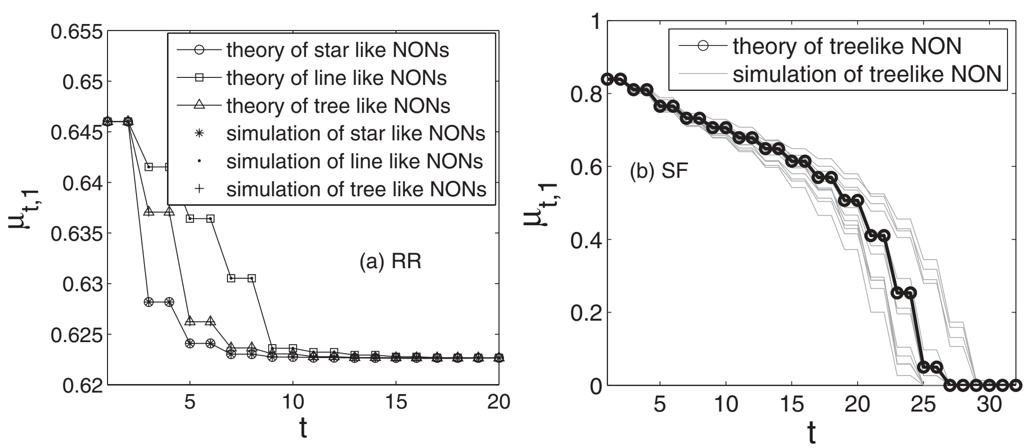

FIG. 5. (a) Simulation results for giant component of root network $\mu_{t,1}$ after t cascading failures for three types of NON composed of 5 RR networks. The structures of the NON are as shown in Fig. 2. For each network in the NON, $N = 100\,000$ and $k = 5$. The chosen value of p is $p = 0.65$, and the predicated threshold $p_c = 0.6047$ [from Eqs. (58) and (62)]. The points are the results of averaging over 40 realizations. It is seen that, while the dynamics is different for the three topologies, the final $P_\infty \equiv \mu_{\infty,1}$ is the same (i.e., the final P_∞ does not depend on the topology of the NON). (b) Simulation results of the giant component of the root network $\mu_{t,1}$ after t cascading failures for tree-like NON composed of the 5 SF networks shown in Fig. 2. For each network in the NON, $N = 100\,000$, $\lambda = 2.3$, and $m = 2$. The value of p chosen is $p = 0.875$ (below p_c). The figure shows 50 simulated realizations of the giant component left after t stages of the cascading failures compared with the theoretical prediction of Eqs. (65), (22) and (23).

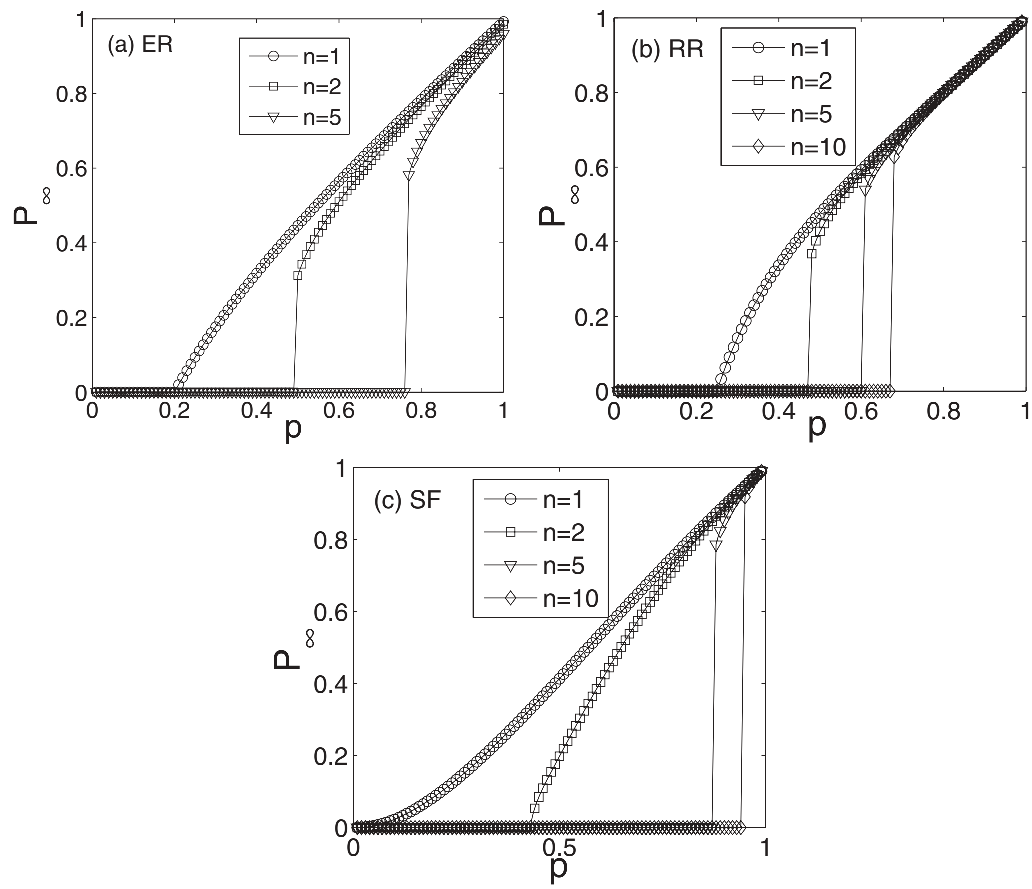

FIG. 6. Loopless NON is composed of (a) ER networks, (b) RR networks, and (c) SF networks. Plotted is P_∞ as a function of p for $k = 5$ (for ER and RR networks) and $\lambda = 2.3$ for SF networks and several values of n. The results obtained using Eq. (43) for ER networks, Eq. (60) for RR networks, and Eq. (65) for SF networks, agree well with simulations.

can be solved explicitly [29]. In this case, the generating functions of the n networks are defined by Eq. (8), $H_i(z_i) = G_i(z_i)$. Using Eq. (30) we obtain $F_i(z_i) = 1 - z_i$ and hence $z_i = z_j \equiv z$.

Using Eqs. (30), (32), and (8) we get

$$P_\infty = 1 - z. \tag{38}$$

and

$$\frac{1}{p} = \frac{\prod_{i=1}^{n}(1 - e^{\bar{k}_i(z-1)})}{1 - z}. \tag{39}$$

Hence the mutual giant component satisfies the self-consistent equation

$$P_\infty = p\prod_{i=1}^{n}(1 - e^{-\bar{k}_i P_\infty}). \tag{40}$$

The value of mutual giant component at criticality, P_∞^c satisfies [see Eqs. (30)–(33)]

$$\sum_{i=1}^{n}\frac{\bar{k}_i e^{-\bar{k}_i P_\infty^c}}{1 - e^{-\bar{k}_i P_\infty^c}} = \frac{1}{P_\infty^c}, \tag{41}$$

and hence

$$p_c = \frac{P_\infty^c}{\prod_{i=1}^{n}(1 - e^{-\bar{k}_i P_\infty^c})}. \tag{42}$$

B. Case of network of networks with same average degree

When the n networks have the same average degree $\bar{k}$, $\bar{k}_i = \bar{k}$ $(i = 1, 2, \ldots, n)$, Eq. (40) gives the percolation law for the order parameter as function of $\bar{k}$, p, and n [29]:

$$P_\infty = p[1 - \exp(-\bar{k}P_\infty)]^n. \tag{43}$$

The solutions of equation (43) for several n values are shown in Fig. 6(a) and for several $\bar{k}$ values are shown in Fig. 7(a). Results are in excellent agreement with simulations. The special case $n = 1$ is the known ER second-order percolation law for a single network [31–33]. The giant component at criticality satisfies simplified Eq. (41):

$$e^{-\bar{k}P_\infty^c} = \left[\bar{k}nP_\infty^c + 1\right]^{-1}. \tag{44}$$

If we introduce a new parameter $w = -\bar{k}P_\infty^c - 1/n$, the solution of Eq. (44) can be expressed in terms of the Lambert function $W(w)$:

$$w = W_{-}[-1/n\exp(-1/n)], \tag{45}$$

where $W_{-}(x)$ is the smallest of the two real roots of the Lambert equation

$$\exp(W_{-})W_{-} = x. \tag{46}$$

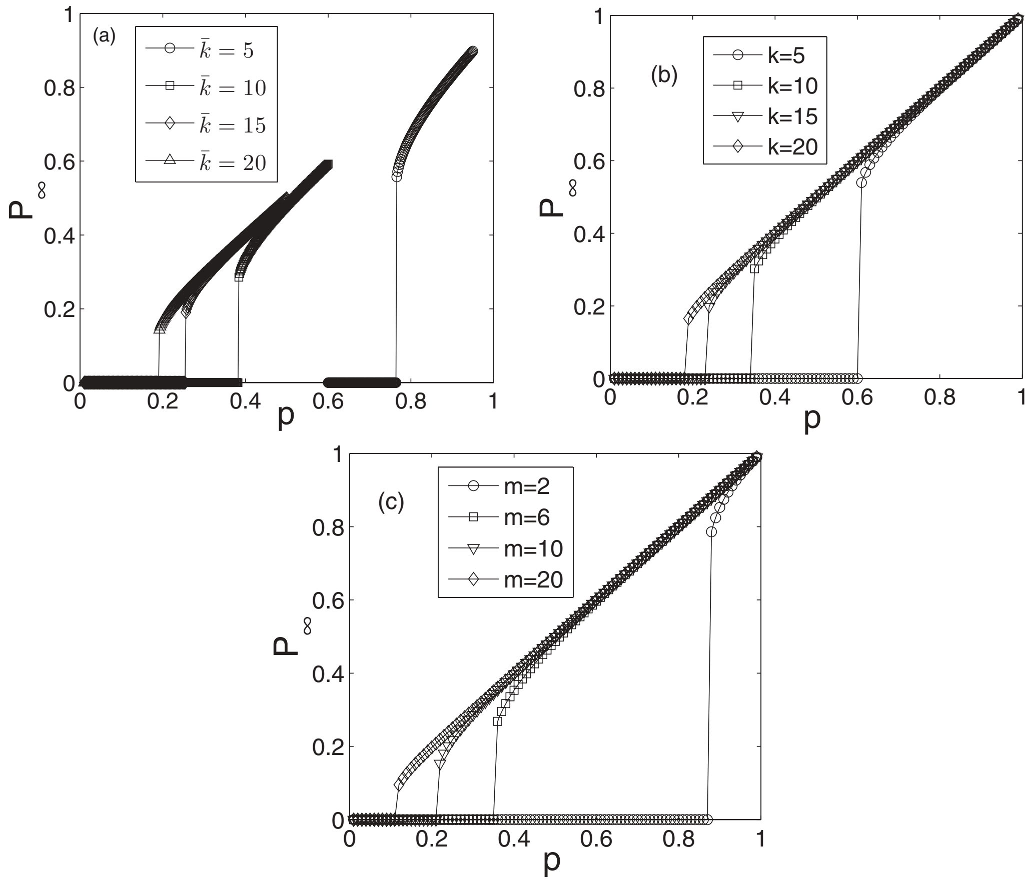

FIG. 7. Loopless NON is composed of (a) ER networks, (b) RR networks, and (c) SF networks. Plotted is P_∞ as a function of p for $n = 5$ for several values of $\bar{k}$ (ER networks), k (RR networks), and several values of m (SF networks for $\lambda = 2.3$). The results obtained using Eq. (43) for ER networks, Eq. (60) for RR networks, and Eq. (65) for SF networks, agree well with simulations.

(The largest root in this case is a trivial solution $W_+ = -1/n$.) Thus we obtain p_c and $P_\infty(p_c)$ by substituting $\bar{k}_i = \bar{k}$ into Eqs. (41) and (38), with the result

$$p_c = -\frac{w}{\bar{k}[1 + 1/(nw)]^{n-1}}, \tag{47}$$

and

$$P_\infty(p_c) = -(w + 1/n)/\bar{k}. \tag{48}$$

For $n = 1$ we obtain the known ER results $p_c = 1/\bar{k}$, and $P_\infty = 0$ at p_c (representing the second-order transition)

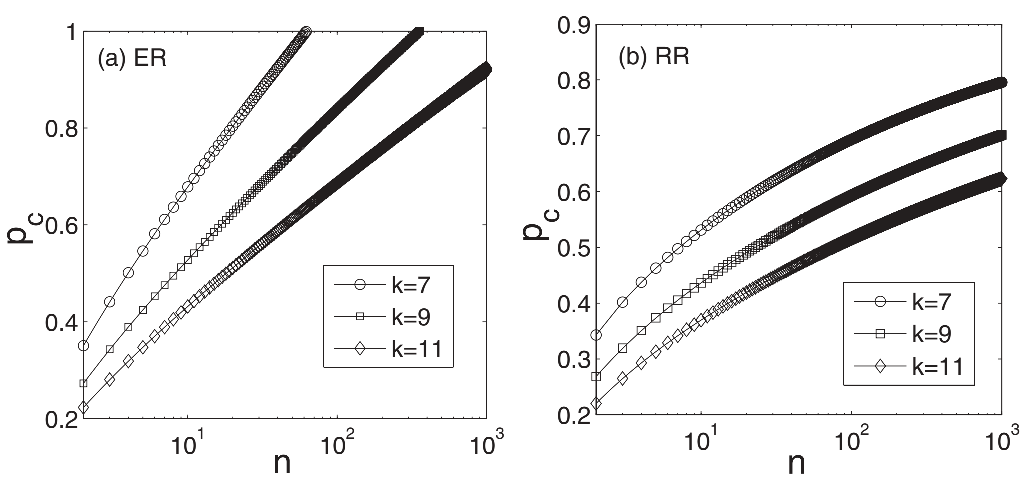

FIG. 8. Critical fraction p_c for different k and n for (a) an ER NON system and (b) a RR NON system. The results for the ER NON system are obtained from Eqs. (44) and (47), while the results of the RR NON system are obtained from the solution of Eqs. (58) and (62). The results are in good agreement with simulations. In the simulations p_c was calculated from the number of cascading failures which diverge at p_c [39] (see also Fig. 10).

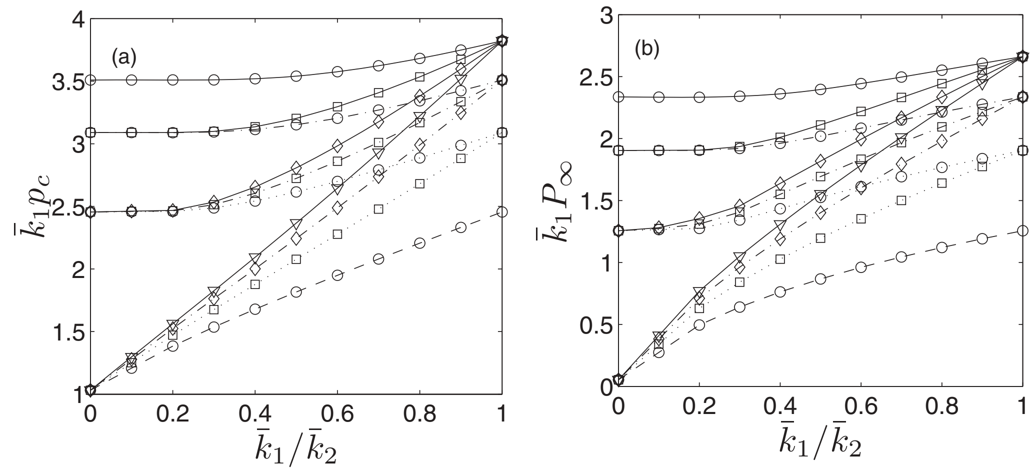

FIG. 9. For a loopless network of n ER networks, (a) $\bar{k}_1 p_c$ and (b) $\bar{k}_1 P_\infty$ as function of the ratio $\bar{k}_1/\bar{k}_2$ for $n = 2$ (dashed), $n = 3$ (dotted), $n = 4$ (dashed dotted), and $n = 5$ (solid) and for $s = 1$ (○), $s = 2$ (□), $s = 3$ (◇), and $s = 4$ (▷), where s denotes the number of individual networks whose average degree are the same $\bar{k}_2$ and average degree of the other $n - s$ networks are $\bar{k}_1$. The results are obtained using Eqs. (38), (47), (48), and (53).

[31–33]. Substituting $n = 2$ in Eqs. (47) and (48) we obtain the exact results derived in Ref. [6].

Since for ER networks $P(0) + P(1) > 0$ for any $\bar{k}$, the NON consisting of n ER networks for sufficiently large n must completely disintegrate even for $p = 1$ [see Fig. 8(a)]. This occurs as the right-hand side of Eq. (47) becomes greater than unity. Conversely, for each n there exists $\bar{k}_{\min}(n)$, such that for $\bar{k} < \bar{k}_{\min}(n)$ the network of n ER networks completely disintegrates even for $p = 1$. Substituting $p_c = 1$ into Eq. (47) we obtain $\bar{k}_{\min}(n)$ as function of n:

$$\bar{k}_{\min}(n) = -\frac{w}{[1 + 1/(nw)]^{n-1}}. \tag{49}$$

Since for $n \to \infty$,

$$W_-\left[\frac{-1}{n}\exp\left(\frac{-1}{n}\right)\right] = -\ln n + O(\ln n), \tag{50}$$

we have

$$\bar{k}_{\min}(n) = \ln n + O(\ln n). \tag{51}$$

Note that Eq. (49) together with Eq. (44) yield the value of $\bar{k}_{\min}(1) = 1$ for $n = 1$, reproducing the known ER result; that $\langle k\rangle = 1$ is the minimum average degree needed to have a giant component. For $n = 2$, Eq. (49) yields the result obtained in Ref. [6], namely,

$$\bar{k}_{\min} = 2.4554. \tag{52}$$

In contrast, for a NON of n RR networks, $P(0) + P(1) = 0$, and hence p_c approaches 1 only when $n \to \infty$ [see Fig. 8(b)] and $k_{\min}(n) = 2$ for any n (see also Sec. V).

C. Case of network of networks with two different average degrees

We next study the case where the average degree of all n networks is not the same. Without loss of generality we assume that s networks have the same average degree $\bar{k}_2$, and other $n - s$ networks have the same average degree $\bar{k}_1$. We define $\alpha \equiv \bar{k}_1/\bar{k}_2$ where $0 < \alpha \leqslant 1$. Using Eq. (39) we can show that $f_c \equiv e^{\bar{k}_1(z_c-1)}$ satisfies

$$f_c = \exp\left[\frac{(f_c - 1)\left(1 - f_c^{1/\alpha}\right)}{(n - s)f_c\left(1 - f_c^{1/\alpha}\right) + s f_c^{1/\alpha}(1 - f_c)/\alpha}\right]. \tag{53}$$

Results for p_c and the mutual giant component for different values of s and α are shown in Fig. 9. The case of $\bar{k}_1 \ll \bar{k}_2$ is interesting. In this limit, the s networks with large $\bar{k}_2$, due to their good connectivity, cannot cause further damage to the $n - s$ networks with $\bar{k}_1$. Thus the NON can be regarded, with respect to percolation, as a NON of only $n - s$ networks. Indeed, for $\alpha \to 0$, P_∞ as a function of p, s, n, and $\bar{k}_1$ is described by Eq. (43), but n and $\bar{k}$ are replaced by $n - s$ and $\bar{k}_1$, respectively, as seen also in Fig. 9.

Our analytical results are in very good agreement with simulations. To determine p_c in simulations we measure the

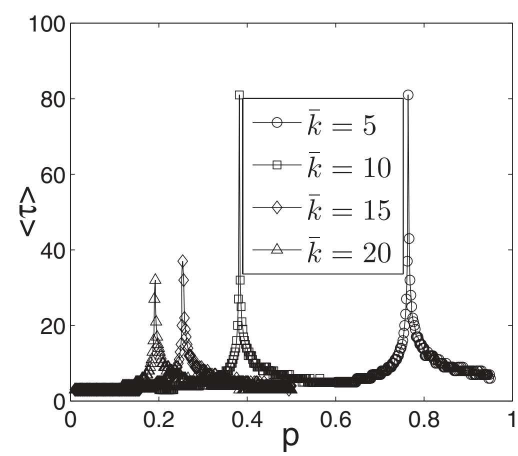

FIG. 10. For a star-like network of 5 ER networks, the average convergence stage $\langle\tau\rangle$ is plotted as a function of p for different $\bar{k}$. In the simulations $N = 10^6$, and averages are obtained from 30 realizations. This feature enables us to find an accurate estimate for p_c in simulations [39].

number of cascading failures (iterations) until the system reaches a steady state (see Fig. 10). As found in Ref. [39], when cascading failures occur, near criticality, their number diverges. Thus the value of p_c where the peak occurs can be used as a good estimate for p_c. This is analogous to the case of the second-order phase transition of regular percolation, in which the size of the second largest cluster diverges at p_c [40].

V. ANALYTICAL RESULTS FOR CASE OF NETWORK OF NETWORKS COMPOSED OF N RR NETWORKS

Next, we study the case of a tree-like NON of n random regular (RR) networks. The degree of network i is k_i. Using Eqs. (1) and (3), we obtain,

$$G_i(z_i) = (z_i)^{k_i}, \tag{54}$$

$$H_i(z_i) = (z_i)^{k_i-1}. \tag{55}$$

Substituting Eqs. (54) and (55) in Eqs. (29) and (30) we obtain equations

$$\frac{1}{p} = \frac{\left(1 - z_i^{k_i-1}\right)\prod_{j=1,j\neq i}^{n}\left(1 - z_j^{k_j}\right)}{1 - z_i}, \tag{56}$$

and

$$P_\infty = p\prod_{j=1}^{n}\left(1 - z_j^{k_j}\right). \tag{57}$$

When all n networks have the same degree k, i.e., $k_i = k$ $(i = 1, 2, \ldots, n)$, $z_i = z$, and the n equations [Eqs. (56)] are reduced to a single equation

$$\frac{1}{p} = \frac{(1 - z^{k-1})(1 - z^k)^{n-1}}{1 - z} \equiv R(z), \tag{58}$$

which can be solved graphically for any p (see Fig. 11). The fraction of nodes in the mutual giant component is

$$P_\infty = p(1 - z^k)^n. \tag{59}$$

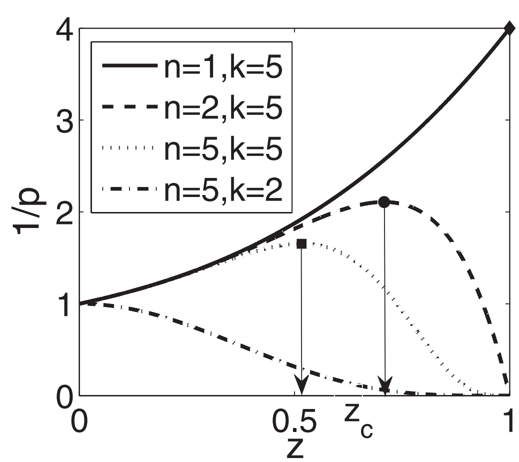

FIG. 11. For the RR NON, $1/p$ as a function of z for different values of k and n. All the lines are produced using Eq. (56). The symbols $\diamond$, $\circ$, and $\square$ show for $k = 5$ the critical solutions for $n = 1$, $n = 2$, and $n = 5$ respectively. These critical thresholds coincide with the results in Fig. 7(b). The dashed-dotted line shows that, when $k = 2$, the function (56) has only a trivial solution $z = 0$ for $p = 1$.

We can obtain P_∞ as a function of p by substituting z from Eq. (58) into Eq. (57),

$$P_\infty = p\left[1 - \left(p^{\frac{1}{n}}P_\infty^{\frac{n-1}{n}}\left\{\left[1 - \left(\frac{P_\infty}{p}\right)^{\frac{1}{n}}\right]^{\frac{k-1}{k}} - 1\right\} + 1\right)^{k}\right]^{n}. \tag{60}$$

When $k = 2$, Eq. (58) has only a trivial solution $z = 0$ for any n. When $k > 2$ and $n = 1$, $R(z)$ is an increasing function of z, so the critical threshold p_c can be obtained by substituting $z \to 1$ into Eq. (58), $p_c = 1/(k-1)$ and $P_\infty^c = 0$ as known for the second-order percolation phase transition of random regular network. When $k > 2$ and $n \geqslant 2$, the critical case corresponds to the maximal $R(z)$, as shown in Fig. 11. Thus, we obtain that the value of z_c satisfies

$$1 = (1 - z_c)z_c^{k-2}\left[\frac{(n-1)kz_c}{1 - z_c^k} + \frac{k-1}{1 - z_c^{k-1}}\right]. \tag{61}$$

Solving z_c from Eq. (61), we obtain the critical value of p_c:

$$p_c = \frac{z_c - 1}{\left(z_c^{k-1} - 1\right)\left(1 - z_c^k\right)^{n-1}}. \tag{62}$$

The numerical solutions of p_c increasing as a function of n are shown in Fig. 8(b). The numerical solutions of Eq. (60) are shown in Figs. 6(b) and 7(b). Here, again, like in the ER case, for $n = 1$ we obtain the known continuous second-order percolation transition, while for $n > 1$ we obtain discontinuous, first-order transitions. In contrast to ER case, for RR NON, $p_c < 1$ for any n, because for a RR network $P(0) + P(1) = 0$. Accordingly, ER NON is significantly more vulnerable compared to RR NON, due to the critical role played in the ER by singly connected nodes. Since $\lim_{n\to\infty} p_c = 1$, even RR NONs with large k become extremely vulnerable as $n \to \infty$. Indeed, solving Eqs. (61) and (62) in the limit $n \to \infty$ we obtain [41]

$$p_c = 1 - \left(1 + \frac{1}{k}\right)(kn)^{-\frac{1}{k-1}} + O\left(n^{-\frac{1}{k-1}}\right). \tag{63}$$

VI. ANALYTICAL RESULTS FOR A NETWORK OF NETWORK COMPOSED OF N SCALE-FREE NETWORKS

Next we study the case of a tree-like NON composed of n scale-free (SF) networks. SF networks are characterized by a power law degree distribution, $P(k) \sim k^{-\lambda}$ with $m \leqslant k \leqslant M$, where m is the minimal degree and M is the maximal degree of a node. Reference [42] shows that it is possible to construct an uncorrelated SF network without multiple and looped links only if $M \ll \sqrt{N}$, a natural structural cutoff. However, in random SF networks, without an imposed cutoff an expected natural cutoff for M scales as $N^{1/(\lambda-1)} \geqslant \sqrt{N}$ for $\lambda \leqslant 3$ [11]. Thus, the generating function method produces correct results in thermodynamic limit only if M increases with the network size slower than $\sqrt{N}$. However, for interdependent SF networks with $\lambda > 2$, the general formalism of Eqs. (30), (32), and (33) remains unchanged. This is true because the only singularity that can affect the behavior of these equations

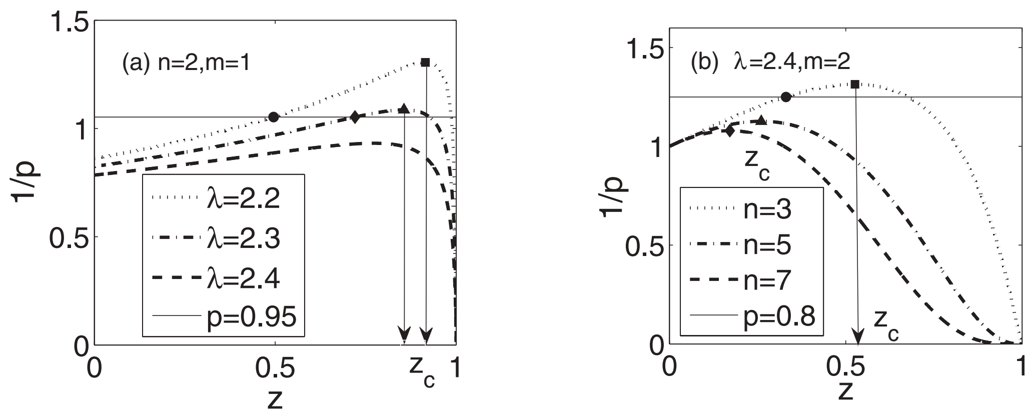

FIG. 12. For the SF NON, $1/p$ as a function of z (a) for different values of λ when $n=2$ and $m=1$, (b) for different values of n when $\lambda=2.4$ and $m=2$. All the lines are produced from the $R_{i,i,n}(z)$ function, Eq. (68). (a) The symbols $\circ$ and $\diamond$ show the physical solutions for $\lambda=2.2$ and $\lambda=2.3$ respectively when $p=0.95$. The symbols $\Box$ and $\triangleright$ show the critical solutions (z_c,p_c) for $\lambda=2.2$ and $\lambda=2.3$ when $p=p_c$. (b) The symbol $\circ$ shows the physical solution for $n=3$, $p=0.8$. The critical solutions (z_c,p_c) are shown as $n=7$, ($\diamond$), $n=5$ ($\Box$), and $n=3$ ($\triangleright$).

is the factor

$$1-H(z)\sim(1-z)^{\lambda-2}. \tag{64}$$

Hence, $\lim_{z_i\to 1}F_i(z_i)=0$ and $\lim_{z_i\to 1}R_i(z_i)=0$ as for any other NON with a finite second moment of the degree distribution (Fig. 12). The critical threshold is determined by the maximum of $R_i(z_i)$ which is achieved for $z_i<1$, for which the tail of the degree distribution is not important for accurate calculation of generating functions. Hence we expect that the analytical results based on Eqs. (30) and (32) are correct in the thermodynamic limit (33) for SF networks with and without a structural cutoff. In summary, we do not expect any qualitative differences in the behavior of SF NONs compared to NONs with a finite second moment. The value of the lower cutoff m is more important than λ and M for the behavior of SF NONs for $n\to\infty$. If $m=1$, $P_i(1)>0$, and hence such networks completely disintegrate for sufficiently large n even if $p=1$ (see Sec. VII).

In order to test our analytical predictions, we analyze a NON of n SF networks with the degree distribution defined by the generating function

$$G_i(z_i)=\frac{\sum_m^M[(k+1)^{1-\lambda_i}-k^{1-\lambda_i}]z_i^k}{(M+1)^{1-\lambda_i}-m^{1-\lambda_i}}, \tag{65}$$

with the same λ, m, and M for all networks. Figs. 6(c) and 7(c) show the solutions for P_∞ for several values of n and m, respectively. For $n\geqslant 2$, p_c becomes finite compared with the case of $n=1$ where $p_c=0$ [11]. Note that, when the SF are partially dependent, $p_c=0$ even for coupled networks [43].

VII. CRITICAL EFFECT OF SINGLY CONNECTED NODES

An interesting question is how vulnerable the NON becomes for $n\to\infty$. Will it for sufficiently large n collapse even for $p=1$? For any n, does there exist $p(n)<1$ such that, for $1>p>p(n)$, the NON has a nonzero mutual giant component? We will show that such a $p(n)$ exists if and only if $P_j(0)+P_j(1)=0$ (condition I), provided that all the networks of the NON have a finite fraction of nodes of finite degree; namely, that there exist constants $\eta>0$ and $M\geqslant 0$ such that for any j

$$\sum_{k=0}^{M}P_j(k)\geqslant\eta \tag{66}$$

(condition II).

In order to show this, we must first show that equations (31) $F_j(z_j)=F_i(z_i)$ define a unique monotonically increasing function $z_j=F_j^{-1}(F_i(z_i))$ for $z_i\in[0,1]$ if $F_i(0)\leqslant F_j(0)$ or for $z_i\in[z_{ij},1]$, where $F_i(z_{ij})=F_j(0)$ if $F_i(0)>F_j(0)$. In order to prove this, we must show that $F_i(z_i)$ is a monotonically decreasing function for $z_i\in[0,1]$. This will follow from the monotonical increase of the function $[1-H_i(z)]/(1-z)$ for $z\in[0,1)$, because $1-G_i(z)$ monotonically decreases. Indeed,

$$\frac{d}{dz}\frac{1-H_i(z)}{1-z}=\frac{[1-H_i(z)-(1-z)H_i'(z)}{(1-z)^2}=H_i''(\theta)/2>0, \tag{67}$$

where $0<\theta<1$, which follows from the Taylor expansion formula with a residual term since $H_i''(\theta)>0$ for any degree distribution except the trivial cases $P_i(m)=0$ for $m>1$, for which the giant component does not exist even for an isolated network. Thus, $F_i(z)$ monotonically decreases.

We next show that if $P_j(0)+P_j(1)=0$ for all $1\leqslant j\leqslant n$ the NON has a nonzero giant component for any n provided that $\bar{k}_j>2$ for each network. The last condition excludes degenerate loop-like networks for which each node has degree 2. Among all j we can select $j=i$, such that $F_i(0)\leqslant F_j(0)$. Then for any $z_i\in[0,1]$ we have $1\geqslant z_j(z_i)\geqslant 0$ and we can represent Eq. (32) as

$$\begin{aligned}R_{i,\ell,n}(z_i)&=R_{i,i,n}(z_i)\\&=\frac{\prod_{j=1,j\neq\ell}^{n}[1-G_j(z_j(z_i))][1-H_\ell(z_\ell(z_i))]}{1-z_\ell(z_i)}=\frac{1}{p}.\end{aligned} \tag{68}$$

If $P_j(0)+P_j(1)=0$ for any j then $H_j(0)=G_j(0)=0$. Hence, $F_i(0)=F_j(0)$ for any i and j and we can select any i, so that $z_j(z_i)|_{z_i=0} = 0$ and $R_{i,\ell}(0) = 1$. Moreover,

$$F_i'(0) = -1 + \frac{2P_i(2)}{\bar{k}_i} < 0 \tag{69}$$

and

$$\left.\frac{dz_j}{dz_i}\right|_{z_i=0} = \left(1 - \frac{2P_i\,(2)}{\bar{k}_i}\right) \Big/ \left(1 - \frac{2P_j\,(2)}{\bar{k}_j}\right) > 1 - \frac{2P_i\,(2)}{\bar{k}_i}$$
$$= C_1 > 0. \tag{70}$$

Finally,

$$\left.\frac{dR_{i,i,n}\,(z_i)}{dz_i}\right|_{z_i=0} = 1 - \frac{2P_i\,(2)}{\bar{k}_i} > 0. \tag{71}$$

Hence $R_{i,\ell}(z_i)$ must reach its maximum value $r = R_{i,\ell}(z_{i,m}) > 1$ at $z_i = z_{i,m}$ where $0 < z_{i,m} < 1$, for any n. Accordingly, Eq. (68) has a nontrivial solution for $1/r < p < 1$, and the NON has a nonzero mutual giant component for sufficiently large p for any n.

Next we show that if $P_j(0) + P_j(1) = 0$ for all $j = 1, 2, \ldots, n$, then a NON has a nonzero giant component for any n for $p_c \leqslant p \leqslant 1$, and if condition II is satisfied then $p_c \to 1$ for $n \to \infty$. Suppose for $n = 2$, the maximum of $R_{i,\ell,2}(z_i)$ is r_2. Given $\delta > 0$ we will find $L > 0$ such that for $n > L$, $R_{i,\ell,n} < 1 + \delta$. Because $R_{i,\ell,2}(z_i)$ has finite derivatives, we can select $z_* < C_0\delta$ such that $R_{i,\ell,2}(z_*) < 1 + \delta$. Because $dz_j/dz_i > C_1$, and the second derivative of $z_j(z_i)$ does not exceed a constant C_2 for any j for $z \in [0,z_*]$, it is clear that $z_j(z_*) > C_1 z_* - z_*^2 C_2/2$. Thus, we can select $\tilde{z}_* < z_*$ such that $z_j(\tilde{z}_*) > z_* C_3$. Hence, if condition II is satisfied $1 - G_j(z_j(z_*)) < 1 - \eta(\delta C_0 C_3)^M$ and hence when $n > L = \ln[(1+\delta)/r_2]/\ln[1 - \eta(\delta C_0 C_3)^M] + 2$, $R_{i,\ell,n}(z_i) < 1 + \delta$, for any $z_i \in [0,1]$.

Suppose now that the function $R_{i,\ell,2}(0) < 1$ and condition II is satisfied. In this case for sufficiently large n, $R_{i,\ell,n}(z_i) < 1$ for any $z_i \in [0,1]$, Let again r_2 be the maximum of $R_{i,\ell,n}(z_i)$. Since $R_{i,\ell,2}(z_i)$ is an analytical function, $R_{i,\ell,2}(z_i) < 1$ for $z_i < z_*$. Making analogous considerations as before, we see that for $n > \ln[1/r_2]/\ln[1 - \eta(z_* C_3)^M] + 2$, $R_{i,\ell,n}(z_i) < 1$ for any $z_i \in [0,1]$.

Now we will show that if for a network r, $P_r(0) > 0$ then $R_{i,\ell,2}(0) < 1$. In order to show this we will use identity

$$R_{i,\ell,2}(0) = [1 - H_i(0)][1 - G_\ell(z_\ell(0))]. \tag{72}$$

If $F_\ell(0) = F_i(0)$, we can select $\ell = r$. If $r = \ell$ we have $1 - G_\ell(z_\ell(0)) < 0$. If $r = i$, then $F_i(0) < F_\ell(0)$. Thus $z_\ell(0) > 0$ and hence $1 - G_\ell(z_\ell(0)) < 1$.

Now we will show that if for a network $j = r$, $P_r(1) > 0$, then $R_{i,\ell,2}(0) < 1$. If $F_\ell(0) = F_i(0)$, we select $i = r$. If $r = i$ our proposition follows from Eq. (72) because $1 - H_i(0) < 1$ If $r = \ell$, it means that $F_i(0) < F_\ell(0)$. Thus $z_\ell(0) > 0$ and hence $1 - G_\ell(z_\ell(0)) < 1$.

VIII. CONCLUSION

In summary, we have developed a framework, Eqs. (20) and (21), for studying percolation of NON from which we derived an exact analytical law, Eqs. (43) (for ER networks) and (60) (for RR networks), for percolation in the case of a network of n fully interdependent networks. In particular, we find that, for any $n \geqslant 2$, cascades of failures naturally appear and the phase transition becomes a first-order transition compared to a second-order transition in the classical percolation of a single network. These findings show that percolation theory and graph theory of a single network is a limiting case of a more general case of interdependent networks. Due to cascading failures which increase with n, vulnerability significantly increases with n. We also find that for any tree-like network of networks the critical percolation threshold and the mutual giant component depend only on the number of networks and not on the topology (see Fig. 1). We discuss the cases for n coupled ER networks, RR networks, and SF networks. We find that for ER NON there exists a minimal $\bar{k} \sim \ln n$ below which even a completely intact NON collapses. This occurs because the ER network has isolated and singly connected nodes. In the absence of such nodes (except in the case of a degenerate loop-like network in which all nodes have degree 2) the NON survives for any n for sufficiently large p, as in the case of RR NON.

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