Robustness of a Network of Networks

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Network research has been focused on studying the properties of a single isolated network, which rarely exists. We develop a general analytical framework for studying percolation of n interdependent networks. We illustrate our analytical solutions for three examples: (i) For *any* tree of n fully dependent Erdős-Rényi (ER) networks, each of average degree $\bar{k}$, we find that the giant component is $P_\infty = p[1 - \exp(-\bar{k}P_\infty)]^n$ where $1 - p$ is the initial fraction of removed nodes. This general result coincides for $n = 1$ with the known second-order phase transition for a single network. For any $n > 1$ cascading failures occur and the percolation becomes an abrupt first-order transition. (ii) For a starlike network of n *partially* interdependent ER networks, P_∞ depends also on the topology—in contrast to case (i). (iii) For a looplike network formed by n *partially* dependent ER networks, P_∞ is independent of n.

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In recent years, dramatic advances in the field of complex networks have occurred [1–14]. The internet, airline routes, and electric power grids are all examples of networks whose function relies crucially on the connectivity between the network components. An important property of such systems is their robustness to node failures, studied using percolation theory. Almost all research has been concentrated on the case of a single or isolated network which does not interact with other networks [15]. Recently, based on the motivation that modern infrastructures are becoming significantly more dependent on each other, a system of two coupled interdependent networks has been studied [16]. A fundamental property of interdependent networks is that when nodes in one network fail, they may lead to the failure of dependent nodes in other networks which may cause further damage in the first network and so on, leading to a global cascade of failures. Buldyrev *et al.* [16] developed a framework for analyzing the robustness of two interacting networks subject to such cascading failures. They found that interdependent networks become significantly more vulnerable compared to their noninteracting counterparts. A generalization has been made in Ref. [17] where a more realistic case of a pair of partially interdependent networks has been studied. In this case both interacting networks have certain fractions of completely autonomous nodes whose function does not directly depend on the nodes of the other network.

In many real systems, more than two networks depend on each other. For example, diverse infrastructures are coupled together, such as water and food supply, communications, fuel, financial transactions, and power stations [18–21]. Understanding the robustness due to such interdependencies is one of the major challenges for designing resilient infrastructures.

Here we develop a theory of robustness of a system of n interdependent networks, which can be graphically represented (see Fig. 1) as a network of networks (NON), based on the percolation approach. We develop an exact analytical approach for percolation of a NON system composed of n fully or partially coupled randomly connected networks. Our results generalize the known results for the percolation of a single network ($n = 1$) and the $n = 2$ result found recently [16,17], and show that while for $n = 1$ the percolation transition is a second-order transition, for $n > 1$ cascading failures occur and the transition becomes a first order. Our results for n interdependent networks suggest that the classical percolation theory extensively studied in physics and mathematics is a limiting case of $n = 1$ of a general theory of percolation in a NON. As shown here this general theory has many novel features that are not present in classical percolation theory.

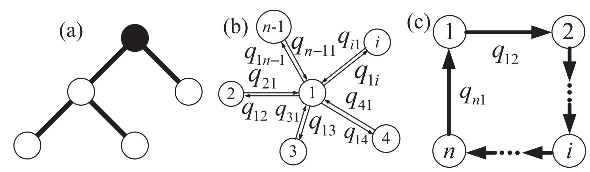

FIG. 1. (a) A treelike NON composed of five fully interdependent networks. (b) Graphical representation of a system of interdependent networks as a starlike NON. Circles represent networks, arrows pointing from network i to network 1 represent partial dependency of network 1 on network i indicating that $q_{i1} > 0$ fraction of nodes in network 1 depend on nodes in network i. (c) A looplike network of networks, where each network depends partially only on one network.

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In our generalization, each node in the NON is a network itself and each link represents a partially dependent pair of networks. We assume that each network i ($i = 1, 2, \ldots, n$) of the NON consists of N_i nodes linked together by connectivity links. Two networks i and j form a partially dependent pair if a certain fraction $q_{ji} > 0$ of nodes of network i directly depend on nodes of network j, i.e., they cannot function if the nodes in network j on which they depend do not function [See Fig. 1(b)]. Dependent pairs are connected by unidirectional dependency links pointing from network j to network i. This convention symbolizes the fact that nodes in network i get supply from nodes in network j of crucial commodity, for example, electric power if network j is a power grid. A partially dependent pair becomes fully dependent if $q_{ij} = q_{ji} = 1$.

We assume that after an attack or failure only a fraction of nodes p_i in each network i remains. We also assume that only nodes which belong to a giant connected component of each network i remain functional. This assumption leads to a cascade of failures: nodes in network i which do not belong to its giant component fail, causing failures of nodes in other networks which depend on the failing nodes of network i. The failure of these nodes causes further failure in network i and so on. Our goal is to find the fraction of nodes $P_{\infty,i}$ of each network which remains functional at the end of the cascade of failures as a function of all p_i and all q_{ij}.

In each particular case, the cascade of failures and the final giant components of the networks can be readily found by computer simulations. However, it is important to find an analytical solution at least under some simplifying assumptions. This analytical solution can serve as a benchmark for simulated solutions of more realistic cases.

In this study we develop an analytical solution of the case in which all networks in the NON are randomly connected networks characterized by degree distribution $P_i(k)$, where k is a degree of a node in network i. We further assume that each node a in network i may depend on only one node b in network j (uniqueness condition) and if node a in network i depends on node b in network j and node b in network j depends on node c in network i, node a must coincide with node c (no-feedback condition) [22].

We will arrive at a system of iterative equations somewhat analogous to Kirchhoff equations for the resistor network. This system of equations has n unknowns x_i which represent the fraction of nodes that survived in the network i after removing all nodes affected by the initial attack and the nodes depending on the failed nodes in other networks. However, x_i does not take into account the further failing of nodes due to the internal connectivity of network i. The final giant component of each network can be found from the equation $P_{\infty,i} = x_i g_i(x_i)$, where $g_i(x_i)$ is the fraction of the remaining nodes of network i which belong to its giant component. The function $g_i(x_i)$ can be expressed [16,23–25] in terms of the generating function $G_i(z) = \sum_k z^k P_i(k)$ of the degree distribution $P_i(k)$ and its normalized derivative $H_i(z) = G_i'(z)/G_i'(1)$ as $g_i(x_i) = 1 - G_i(1 - x_i(1 - f_i))$, where an auxiliary variable f_i satisfies equation $f_i = H_i(1 - x_i(1 - f_i))$.

The unknowns x_i satisfy the system of n equations:

$$x_i = p_i \prod_{j=1}^{K} (q_{ji} y_{ji} g_j(x_j) - q_{ji} + 1), \tag{1}$$

where the product is taken over the K networks interlinked with network i by the partial dependency links and

$$y_{ji} = x_j / [q_{ij} y_{ij} g_i(x_i) - q_{ij} + 1], \tag{2}$$

has the meaning of the fraction of nodes in network j survived after the damage from all the networks connected to network j except from network i is taken into account. The damage from network i must be excluded due to the no-feedback condition. In the absence of the no-feedback condition Eq. (1) becomes much simpler since $y_{ji} = x_j$.

We tested the numerical solutions of Eqs. (1) and (2) for many NONs of different topologies and all analytical results presented below by computer simulations of the cascading failures of small NONs, consisting of $n \leq 10$ interdependent networks, each comprised of $N_i = 10^6$ nodes. In all cases we find excellent agreement between the theory and simulations [26].

Note, that if $n = 2$, Eqs. (2) yield $y_{12} = p_1$, $y_{21} = p_2$ and Eqs. (1) can be simplified: $x_1 = p_1[p_2 q_{21} g_2(x_2) - q_{21} + 1]$, $x_2 = p_2[p_1 q_{12} g_1(x_1) - q_{12} + 1]$ which coincides with Ref. [17].

An interesting simplification can be made for a network of networks having a treelike structure without loops [Fig. 1(a)] in which all connected pairs of networks are fully dependent. Note that the no-feedback condition in this case establishes a one-to-one correspondence between all the nodes in different networks of the NON. Accordingly, random attacks on the individual networks which remove fractions of nodes $1 - p_i$ from each network are equivalent to a single attack on one of the networks which removes $1 - p = 1 - \prod_{i=1}^{n} p_i$ fraction of nodes. In this case, Eqs. (1) and (2) yield $x_i g_i(x_i) = x_j g_j(x_j) = P_\infty$, where P_∞ is the fraction of nodes in the mutual giant component which is the same for all the networks in the tree. Finding y_{ij} one by one starting from the singly connected nodes of the NON, one can show that P_∞ is the product:

$$P_\infty = \prod_{i=1}^{n} p_i g_i(x_i), \tag{3}$$

where each x_i satisfies the equation $x_i = P_\infty / g_i(x_i)$. The system of Eq. (3) defines $n + 1$ unknowns: $P_\infty, x_1, x_2, \ldots, x_n$ as functions of $\{p_i\}$ and the degree distributions $\{P_i(k)\}$.

Next we present three examples which can be solved analytically explicitly: (i) a treelike NON fully dependent,

(ii) a starlike NON partially dependent and (iii) a looplike NON partially dependent [see Fig. 1(c)]. All cases represent different generalizations of percolation theory of a single network. (i) We solve explicitly the case of a treelike NON [Fig. 1(a)] formed by n Erdős-Rényi (ER) [27–29] networks with average degrees $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_i, \ldots, \bar{k}_n$, $p_i = p$ and $q_{ij} = q_{ji} = 1$. We have $G_i(x) = H_i(x) = \exp[\bar{k}_i(x-1)]$ [24]. Accordingly $g_i(x_i) = 1 - \exp[\bar{k}_i x_i(f_i - 1)]$, where $f_i = \exp[\bar{k}_i x_i(f_i - 1)]$ and thus $g_i(x_i) = 1 - f_i$. Using Eq. (3) for x_i we get

$$f_i = \exp\left[-p\bar{k}_i \prod_{j=1}^{n}(1 - f_j)\right], \qquad i = 1, 2, \ldots, n. \tag{4}$$

These equations can be easily solved analytically. They have only a trivial solution ($f_i = 1$) if $p < p_c$, where p_c is the mutual percolation threshold. When the n networks have the same average degree $\bar{k}$, $\bar{k}_i = \bar{k}$ ($i = 1, 2, \ldots, n$), we obtain from Eq. (4) that $f_c \equiv f_i(p_c)$ satisfies

$$f_c = \exp[(f_c - 1)/n f_c], \tag{5}$$

where the solution can be expressed in term of the Lambert function $W(x)$ [30], $f_c = -[nW(-\frac{1}{n}e^{-(1/n)})]^{-1}$.

Once f_c is known, we obtain p_c and $P_{\infty,n} \equiv P_\infty(p_c)$,

$$p_c = [n\bar{k}f_c(1 - f_c)^{(n-1)}]^{-1}, \qquad P_\infty(p_c) = (1 - f_c)/n\bar{k}f_c. \tag{6}$$

For $n = 1$ we obtain the known result $p_c = 1/\bar{k}$ of Erdős and Rényi [27–29]. Substituting $n = 2$ in Eqs. (5) and (6) yields the exact results of [16].

To analyze p_c as a function of n for different values of $\bar{k}$, we find f_c from Eq. (5) and substitute it into Eq. (6) [Fig. 2(a)]. We see that the NON becomes less robust with increasing n or decreasing $\bar{k}$ (p_c increases when n increases or $\bar{k}$ decreases). Furthermore, for a fixed n, when $\bar{k}$ is smaller than a critical number $\bar{k}_{\min}(n)$, $p_c \geq 1$ meaning that for $\bar{k} < \bar{k}_{\min}(n)$, the NON will collapse even if a single node fails. Using Eq. (6) we get $\bar{k}_{\min}$ as a function of n [Fig. 2(b)]:

$$\bar{k}_{\min}(n) = [nf_c(1 - f_c)^{(n-1)}]^{-1}. \tag{7}$$

Note that for $n = 1$, Eq. (7) together with Eq. (5) yield the value of $\bar{k}_{\min}(1) = 1$, reproducing the known ER result, that $\bar{k} = 1$ is the minimum average degree needed to have a giant component. For $n = 2$, Eq. (7) yields the result obtained in [16], i.e., $\bar{k}_{\min}(2) = 2.4554$.

From Eq. (3) we obtain an exact equation for the order parameter, the size of the mutual giant component P_∞ for all values of p, $\bar{k}$, and n:

$$P_\infty = p[1 - \exp(-\bar{k}P_\infty)]^n. \tag{8}$$

Solutions of Eq. (8) which are valid for all tree NON topologies (e.g., line, star), are shown in Fig. 2(c) for several values of n. The special case $n = 1$ is the known ER second-order percolation law for a single network [27–29]. In contrast, for any $n > 1$, the solution of (8) yields first-order percolation, i.e., a discontinuity of P_∞ at p_c [see Eq. (6)]. For a treelike NON fully interdependent which is composed of scale-free (SF) networks we substitute into Eq. (3) the generating functions of SF networks. The numerical solutions are shown in Fig. 3(a). (ii) The partially dependent starlike NON [Fig. 1(b)]. We first remove a fraction of $1 - p$ nodes only in the root network. These damages spread to all other networks, and then back to the root network, back and force. In this case, $y_{i1} = 1$, where the index 1 represents the central network and i

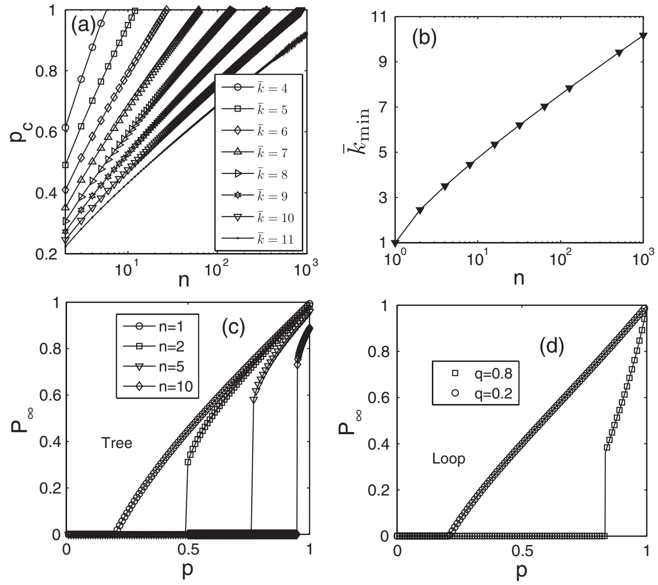

FIG. 2. (a)–(c) Loopless fully coupled ($q = 1$) NON. (a) The critical fraction p_c for different $\bar{k}$ and n and (b) the minimum average degree $\bar{k}_{\min}$ as a function of the number of networks n. The results of (a) and (b) are obtained using Eqs. (6) and (7) respectively. (c) Fraction of nodes in the mutual giant component P_∞ as a function of p for $\bar{k} = 5$ and several values of n, obtained from Eq. (9). (d) The giant component for a loop type partially dependent NON [Fig. 1(c)], P_∞ as a function of p for $\bar{k} = 6$ and two values of q, obtained from Eq. (14).

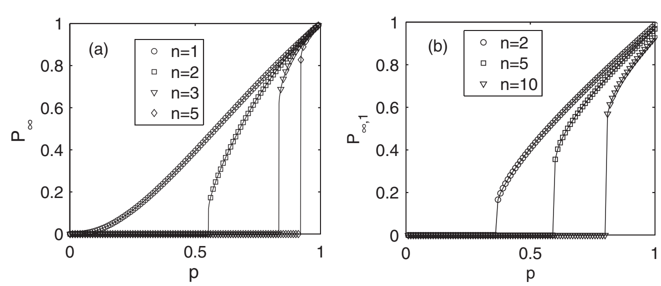

FIG. 3. Dependence of the fraction of nodes at the end of the cascade of failures, P_∞, on the fraction of nodes, p, survived the initial failure for different number of interacting networks, n: (a) for a treelike fully interdependent ($q = 1$) SF NON with $\lambda = 2.5$ and minimum degree 2. (b) for a starlike partially coupled ($q = 0.8$) ER NON with $\bar{k} = 5$.

denotes the other networks. Under the simplifying conditions that $q_{i1} = q_{1i} = q$ and that the average degrees of all networks are equal to $\bar{k}$, we have $x_2 = x_3 = \ldots x_n$, so Eqs. (1) becomes

$$x_1 = p[qg_2(x_2) - q + 1]^{n-1},$$
$$x_2 = pqg_1(x_1)[qg_2(x_2) - q + 1]^{n-2} - q + 1. \quad (9)$$

Since for ER networks $g_i(x_i) = 1 - \exp\{\bar{k}x_i(f_i - 1)\}$ and $f_i = \exp\{\bar{k}x_i(f_i - 1)\}$, from Eqs. (9) we obtain

$$f_1 = \exp\{p\bar{k}(1 - qf_2)^{n-1}(f_1 - 1)\},$$
$$f_2 = \exp\{\bar{k}[pq(1 - f_1)(1 - qf_2)^{n-2} - q + 1](f_2 - 1)\}. \quad (10)$$

From the definitions of $P_{\infty,i}$, g_i and f_i, we obtain

$$P_{\infty,i} = -\ln(f_i)/\bar{k}. \quad (11)$$

Solving numerically Eq. (10), we get f_1 and f_2, and substituting them into Eqs. (11) we get $P_{\infty,1}$ of the root network and $P_{\infty,2}$ of the other networks.

Figure 3(b) shows the solution for the giant component of the root network. Note that for a fully dependent NON, $q = 1$, $f_1 = f_2$ and Eqs. (10) and (11) can be reduced to Eq. (8) as expected. However, while Eq. (8) is derived for $q_{ij} = 1$ and valid for all treelike structures, Eqs. (10) and (11) are derived for $q_{ij} < 1$ and are valid only for a starlike NON. Thus, in contrast to a fully dependent NON, for a partially dependent NON, each tree topology will have a different solution that can be derived from Eqs. (1) and (2). (iii) The case of a looplike NON of n ER networks, shown in Fig. 1(c). In this example all the links are unidirectional, and the no-feedback condition is irrelevant. If the initial attack on each network is the same $1 - p$, $q_{i-1i} = q_{n1} = q$, and $\bar{k}_i = \bar{k}$, using Eq. (1) we obtain that P_∞ satisfies

$$P_\infty = p(1 - e^{-\bar{k}P_\infty})(qP_\infty - q + 1). \quad (12)$$

Note that if $q = 1$, Eq. (12) has only trivial solution $P_\infty = 0$ while for $q = 0$, it yields the known giant component of a single network, as expected. We present numerical solutions of Eq. (13) for two values of q in Fig. 2(d).

In summary, we have developed a general framework, Eqs. (1) and (2), for studying percolation of different types of NONs for any degree distribution. We demonstrate our approach on three examples which can be solved analytically exactly, Eqs. (8), (11), and (12) for three cases of a network of n interdependent ER networks. All these equations represent different generalizations of the known single network case. In particular for any $n \geq 2$, cascades of failures naturally appear for strong coupling, and the phase transition becomes a first-order transition compared to a second-order transition in the classical percolation of a single network ($n = 1$). These findings show that the percolation theory of a single network is a limiting case of a more general case of percolation of interdependent networks. Finally we wish to note that we recently solved the case of a random regular (RR) network of ER networks [31]. Our results show that the percolation threshold and the giant component depend only on the average degree of ER network and the degree of RR network, but not on the number of networks. Thus, the framework presented here opens the possibility to study percolation of different topologies of a NON.

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[1] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[2] R. Albert *et al.*, Nature (London) **406**, 378 (2000).
[3] R. Cohen *et al.*, Phys. Rev. Lett. **85**, 4626 (2000).
[4] D. S. Callaway *et al.*, Phys. Rev. Lett. **85**, 5468 (2000).
[5] R. Albert and A. L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[6] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[7] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW (Physics)* (Oxford University Press, New York, 2003).
[8] C. Song *et al.*, Nature (London) **433**, 392 (2005).
[9] R. P. Satorras and A. Vespignani, *Evolution and Structure of the Internet: A Statistical Physics Approach* (Cambridge University Press, England, 2006).
[10] M. E. J. Newman *et al.*, *Large Scale The Structure and Dynamic Of Networks* (Priceton University Press, Pricenton, NJ, 2006).
[11] G. Caldarelli and A. Vespignani, *Large Scale Structure and Dynamics of Complex Webs* (World Scientific, Singapore, 2007).
[12] A. Barrát *et al.*, *Dynamical Processes on Complex Networks* (Cambridge Univ. Press, England, 2008).
[13] R. Cohen and S. Havlin, *Complex Networks: Structure, Robustness and Function* (Cambridge University Press, England, 2010).
[14] M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, New York, 2010).
[15] Interacting networks were studied independently also by E. A. Leicht and R. M. D' Souza, arXiv:0907.0894. This model is different from the present model and leads to very different results..
[16] S. V. Buldyrev *et al.*, Nature (London) **464**, 1025 (2010).

[17] R. Parshani, S. V. Buldyrev,, and S. Havlin, Phys. Rev. Lett. **105**, 048701 (2010).
[18] J. Peerenboom *et al.*, in Pro. CRIS/DRM/IIIT/NSF Workshop Mitigat. Vulnerab. Crit. Infrastruct. Catastr. Failures (2001).
[19] S. Rinaldi *et al.*, IEEE Control Syst. Mag. **21**, 11 (2001).
[20] V. Rosato *et al.*, Int. J. Crit. Infrastruct. **4**, 63 (2008).
[21] A. Vespignani, Nature (London) **464**, 984 (2010).
[22] The uniqueness condition can be lifted without loss of generality for a NON similar to that in the case of two interacting networks [J. Shao *et al.*, Phys. Rev. E **83**, 036116 (2011).]. The no-feedback condition prevents a fully dependent pair of networks from a complete collapse even without taking into account their internal connectivity.
[23] M. E. J. Newman, S. H Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[24] M. E. J. Newman, Phys. Rev. E **66**, 016128 (2002).
[25] J. Shao *et al.*, Phys. Rev. E **80**, 036105 (2009).
[26] In simulations, we calculate p_c using the fact that the number of cascading failures in an infinite network diverges at p_c, R. Parshani *et al.*, Proc. Natl. Acad. Sci. U.S.A. **108**, 1007 (2010).
[27] P. Erdős and A. I. Rényi, Publ. Math. **6**, 290 (1959).
[28] P. Erdős and A. Rényi, Inst. Hung. Acad. Sci. **5**, 17 (1960).
[29] B. Bollobás, *Random Graphs* (Academic, London, 1985).
[30] R. M. Corless *et al.*, Adv. Comput. Math. **5**, 329 (1996).
[31] J. Gao *et al.*, ‘‘Percolation of a Random Regular Network of Erdős Rényi Networks’’ (to be published).