MICROSCOPIC CALCULATION OF CRITICAL EXPONENTS WITHOUT THE $1/n$ OR ϵ EXPANSIONS

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Critical exponents of a Bose system are calculated microscopically without an expansion in $1/n$ or ϵ. As expected, quantum corrections are found to be absent and the results to agree with the $1/n$ expansion result, for $n = 2$, to $O(1/n)$.

The purpose of this letter is to discuss the critical behavior of a quantum system, which exhibits a phase transition, without the commonly employed techniques of $1/n$ or ϵ expansions [1] (n is the number of components of the order parameter and $\epsilon = 4-d$, where d is the spatial dimension of the system). Our motivations for such an investigation are three-fold: First, to present an alternative calculation of critical indices not based on an expansion in n or d, in order to break away from the *a priori* assumption of "universal" significance given to these quantities; second, to treat a quantum mechanical system strictly within a quantum statistical formulation to test the universality assumption that the critical exponents are independent of quantum effects; third, to establish a closer connection between the standard microscopic approaches to many-body theory [2] and the newly developed $1/n$ and ϵ expansions [1].

As our model, we consider a system of spinless bosons of mass m at temperature T above T_c and at a fixed density in a unit volume. In order to avoid arbitrary assumptions about the strength of the potential and introduction of cutoffs [3, 4], we consider only a specific potential and assume that the particles are interacting with the Coulomb potential, $V(q) = 4\pi e^2/q^2$ and the system is placed in a rigid background of opposite charge to ensure overall charge neutrality.

Certain static critical exponents are defined by the asymptotic form of the relevant correlation functions for small k at T_c. For example, the order parameter correlation function $G(k)$ and (in a charged system) [3] the irreducible density correlation function $\Pi(k)$ for small k at T_c behave like $G(k) \sim k^{-2+\eta}$ and $\Pi(k) \sim k^{|\lambda|}$ if $\lambda < 0$, defining the exponents η and λ which we now proceed to calculate for our model.

We employ the usual diagrammatic perturbation theory techniques [2] and seek the lowest order correction to the properties of the non-interacting system (i.e., ideal Bose gas). The simplest self-consistent approximation is the well-known Hartree-Fock approximation, which for a charged Bose gas in the *static* limit takes the form [2]

$$G^{-1}(k) = -\epsilon(k) - \Sigma(k) + \Sigma(0) - r, \qquad (r = 0 \text{ at } T_c) \tag{1}$$

where

$$\Sigma(k) = \int \frac{d^3p}{(2\pi)^3} [V(p)/1 - V(p)\Pi(p)]\, n(\boldsymbol{p}+\boldsymbol{k}), \tag{2}$$

$$\Pi(k) = \int \frac{d^3p}{(2\pi)^3} [n(\boldsymbol{p}+\boldsymbol{k}) - n(p)]\, [\epsilon(\boldsymbol{p}+\boldsymbol{k}) - \epsilon(p)]^{-1}, \tag{3}$$

$\epsilon(k) = \hbar^2 k^2/2m$, and $n(k)$ is the Bose-Einstein distribution function. Eq. (3) represents the contribution of the simple RPA polarization bubble, which using contour integration is found to be

$$\Pi(k) = \left(\frac{2m}{\hbar^2}\right)^2 (4\pi\beta k)^{-1}\left[\arctan\left(\frac{\hbar^2}{2m}\right)\frac{k}{2r^{1/2}} + \left(\frac{\hbar^2\beta}{2m}\right)^{1/2} k\,\frac{\Gamma(1/2)\zeta(1/2)}{2\pi}\right], \tag{4}$$

Note that at T_c (i.e. when $r = 0$), $\Pi(k)$ diverges as $k \to 0$, and in this limit we recover the classical field approximation result [3, 4], $\Pi(k) = (2m/\hbar^2)(8\beta k)^{-1}$. Once (4) is substituted in (2), it is found that the quantum correction term (bracketed in eq. (4)) does not affect the $k^2 \ln k$ term in $\Sigma(k) - \Sigma(0)$, and in the limit $k \to 0$

$$\Sigma(k) - \Sigma(0) = -\epsilon(k)\eta \ln k + O(k^4), \quad \text{where} \quad \eta = 4/3\pi^2, \tag{5}$$

in complete agreement with the $1/n$ expansion result [3, 4] with $n = 2$ for a Bose system with a two-component (real) order parameter.

In order to calculate the exponent λ, we note that to zeroth order (i.e. for a non-interacting system) λ is obtained from eq. (4), which represents the density propagator for the ideal Bose gas [2]. From (4) we have that $\Pi(k) \sim k^{-1}$ for an ideal Bose gas and as expected $\lambda = -1$ to zeroth order [3]. To find the leading correction to this result we can make use of standard diagrammatic expansion for the density response function and calculate the leading correction to RPA. However, we choose to consider another approach which is considerably simpler. We make use of Landau's quasiparticle picture and assume that as $k \to 0$, at T_c, the interacting Bose gas can be viewed as an ideal gas of particles with a modified energy spectrum $\omega(k) \sim k^{2-\xi}$, where ξ is some small number to be determined. It should be pointed out that $\omega(k)$ is a highly temperature dependent quantity and the Landau picture can be used here only because we need the long wavelength form of $\omega(k)$ at T_c. If we were interested in calculating exponents which are not defined at T_c, then our use of the Landau picture could not have been justified. As usual, the excitation spectrum is determined from the pole of the dynamic temperature Green function [2]

$$G(k, \omega_j) = [i\omega_j - \epsilon(k) - \Sigma(k, \omega_j) + \Sigma(0, 0) - r]^{-1}. \tag{6}$$

where $\omega_j = 2\pi j/\beta$ ($j = 0, \pm 1, \pm 2, \ldots$), and $\Sigma(k, \omega_j)$ is the dynamic Hartree-Fock self-energy [2]. We first let $\omega_j \to \omega(k) - i\gamma(k)$ in (6), where $\omega(k)$ and $\gamma(k)$ are both real. From the pole of (6) we find the real part of the excitation spectrum,

$$\omega(k) = \epsilon(k) + \Sigma(k, \epsilon(k)) - \Sigma(0, 0), \tag{7}$$

at T_c, where Σ denotes the real part of the Hartree-Fock self-energy, and to lowest order we have let $\omega(k) \to \epsilon(k)$ on the rhs of (7). The $k^2 \ln k$ term in (7) is found by expanding the distribution functions in the Σ functions and investigating the individual terms. We find that only one term behaves as $k^2 \ln k$, so that (7) becomes

$$\omega(k) = \epsilon(k)[1 - \xi \ln k] \sim k^{2-\xi}, \tag{8}$$

where $\xi = 16/3\pi^2$. In order to find λ to first order we replace all the $\omega(k)$ in (3) with $\epsilon(k)$ and investigate the long wavelength form of $\Pi(k)$. We find

$$\Pi(k) \sim k^{-1+2\xi} \quad \text{or} \quad \lambda = -1 + 32/3\pi^2, \tag{9}$$

in complete agreement with the $1/n$ expansion result for a Bose system [3, 4].

Other exponents can be obtained similarly or from η and λ via the scaling laws [3], e.g. $\gamma = -2(2-\eta)/\lambda - d$ and $-\alpha = \lambda\gamma/2 - \eta$. The results thus obtained are exactly the same as the $1/n$ expansion with $n = 2$ to $O(1/n)$ [3, 4]. We have thus found that for our particular quantum mechanical model the critical exponents are independent of quantum mechanics and do not depend on the details of the interaction. Thus the exponents obtained apply universally to any Bose system, as previously argued by Ma [3].

The significance of the the present calculation is that with standard perturbation theory, i.e. with an expansion essentially in $V(k)$, we obtained the same result as the $1/n$ expansion for our model. Our procedure differs from

both the classical field and quantum versions of the $1/n$ expansion [3, 4] and ϵ expansion [1] in two important ways: (1) we do not *assume* an expansion in n or d, and (2) we are not required to either renormalize the bare potential, as done in the ϵ expansion [1], or to require the screened potential to be given by $-1/\Pi$, as is assumed in $1/n$ expansion [3, 4]. In all previous $1/n$ expansion theories [3, 4] it had to be assumed that the screened potential approaches $-1/\Pi$ in order to eliminate the bare potential ‡ from the theory not by a renormalization procedure but by a screening approximation in the limit $k \to 0$. In the present model calculation we used the unrenormalized perturbation theory, made the usual screening approximation for the Coulomb potential [2], and did not assume that the screened potential is independent of the bare potential. This allowed us to keep the bare interaction in the formulation to the end of the calculation and thus demonstrate that critical exponents are independent of the bare coupling constant of the model (in this case e^2), as well as quantum mechanics.

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‡ The bare potential was also assumed to be weak, i.e., of the order of $1/n$.

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