Percolation model for growth rates of aggregates and its application for business firm growth

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Motivated by recent empirical studies of business firm growth, we develop a dynamic percolation model which captures some of the features of the economical system—i.e., merging and splitting of business firms—represented as aggregates on a d-dimensional lattice. We find the steady-state distribution of the aggregate size and explore how this distribution depends on the model parameters. We find that at the critical threshold, the standard deviation of the aggregate growth rates, σ, increases with aggregate size S as $\sigma \sim S^{\beta}$, where β can be explained in terms of the connectedness length exponent ν and the fractal dimension d_f, with $\beta = 1/(2\nu d_f) \approx 0.20$ for $d=2$ and 0.125 for $d \to \infty$. The distributions of aggregate growth rates have a sharp peak at the center and pronounced wings extending over many standard deviations, giving the distribution a tent-shape form—the Laplace distribution. The distributions for different aggregate sizes scaled by their standard deviations collapse onto the same curve.

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I. INTRODUCTION

The distributions of the size and growth rates of business firms is an important topic in economics [1–39]. The size distribution of the firms can be characterized by a power law $P_S(S) \sim S^{-\tau}$ [40–42], the standard deviation of firm growth rates scales with firm size as a power law with a scaling exponent of approximately $-1/6$, and the distribution of growth rates conditional on firm size collapses onto a Laplace distribution [43–47].

The empirical findings about the statistical properties of firm growth merit an explanation, and the challenge is how to go about doing so. Coase [3,48,49] introduced the dominant framework now used in economics to understand business size. While his discussion of the problem was static, he suggested a framework for introducing dynamics [3]. He suggested that economic activity can be conceived of as a set of concentric circles, with each ring representing an industry. Firms could start out in one industry and expand until no further opportunities are available and then expand into an adjacent industry. The size of the firm would be represented by the area in the circles it covers; the growth of firms would be represented by the change in size as firms went about a process of exploring new activities to undertake.

The approach followed here is to start with this broad framework suggested by Coase and to then translate it into the types of models physicists have used to understand critical phenomena. The power-law behavior observed in the data on firm growth is the hallmark of critical phenomena [50]. While critical phenomena are not entirely absent from the economics literature [51,52], economists in general do not look for evidence of them and conventional economic models do not seek to explain either their existence or their properties.

Using models from physics to capture this conceptual framework poses a number of problems, some mechanical and some more broadly conceptual. Not surprisingly, the former are easier to solve. The set of concentric circles Coase used to illustrate his ideas imply a two-dimensional space of activity. Coase did not provide a mathematical representation of his framework; nor has anyone since. A lattice provides a more natural framework for understanding two-dimensional (2D) (and higher-dimensional) spaces, which is the standard approach in physics. Within this lattice framework, each row can represent a product and each column can represent a consumer. A cell is a (potential) transaction—i.e., the sale of a specific product to a specific customer. We can then think of firms as collections of cells, with the size of the firm being the number of cells. As a firm expands along a row, it sells a particular product to more customers [53,54]. As it expands along a column, it sells more products to the same customer.

The broad conceptual question is what the lattice dimension represents. The industrial dimension in particular should not be understood as a measurable, physical quantity. Rather, modeling economic activity in this fashion captures the notion that each industry is more closely related to some other industries than it is to others. The implication is that whenever a firm operating in one industry chooses to diversify into another, it has natural candidates to consider. As an example, in the 1980s, two tobacco companies, Phillip Morris and RJ Reynolds, found that their cigarette businesses generated more cash than could profitably be reinvested in the cigarette business. As a result, they both sought to move into new lines of business. Both chose food companies: Phillip Morris purchased General Foods and RJ Reynolds purchased Nabisco. Neither went into the petroleum, pharmaceutical, or computer software industries. The similarity of the choices likely was not a coincidence. Thus, without being specific about what makes cigarettes and food processing close to each other, there must be some sense in which they are.

While, as a rule, firms seem to diversify into industries that are related to their existing businesses in some way, there remain some conglomerate firms that operate in a wide variety of industries. A prominent example is General Elec-

tric which, for example, operates the television network NBC and builds jet engines. Ultimately, we would like to understand how it is possible for such firms to evolve. Without a model in which some industries are far removed from others, it is not possible to address the question of how firms evolve in such a way that some firms are engaged in diverse activities.

The model described below is a step in adapting formal models from statistical physics to capture the essential features of Coase's theory. It successfully reproduces some of the key statistical properties of firm size and growth. Specifically, we show that if a dynamic percolation model is tuned to its critical state, the distribution of aggregate sizes is power law, mean growth rates are independent of firm size, and the distribution of growth rates conditional on initial size collapse onto a single curve. The results do differ from the empirical findings in some important respects. While the standard deviation of firm growth rates scales with firm size, the scaling exponent is positive rather than the $-1/6$ found in the empirical data. The model does not, however, embody all of the essential features of Coase's theory. We believe that this is a first step toward a richer model that captures the spirit of Coase and more accurately reproduces the empirical data.

As a first step to understand the growth rates of business firms in the more complicated model, a comparison with the growth rates of aggregates in a standard dynamic percolation model is warranted. However, the distribution of the growth rates of aggregates has not previously been studied. Thus, the primary purpose of this paper is to establish a benchmark set of properties for the growth rates of aggregates in a standard model of dynamic percolation.

II. THE MODEL

Here we introduce a variant of a dynamic percolation model [55–62]. The model is based on a d-dimensional periodic L^d hypercubic lattice (L is the number of cells on one side of the lattice). Each cell has 3^d-1 other cells which are adjacent to this cell at least by one point.

Following Refs. [55,57], we define two probabilities $B \in (0,1)$ and $D \in (0,1)$. B is the probability for each inactive cell to become active ("birth"), and D is the probability for each active cell to become inactive ("death") during one time step of the model. At time $t=0$, each cell of the lattice is assigned to be active with probability $B/(B+D)$ and inactive with probability $D/(B+D)$. Note that this initial condition corresponds to the steady state of the model, so the fraction of active cells remains $\approx B/(B+D)$ at any time step t. At $t=0$, aggregates are defined to be equivalent to active cells. Thus initially the size of each aggregate is equal to unity. As time progresses, aggregates can merge, split, appear, and disappear according to the rules of the model. Note that the aggregate in our model is not equivalent to a static percolation cluster. A percolation cluster is defined as an equivalency class of all active cells connected via paths that go from each active cell to all its active neighbors. In our model, the aggregate must be connected, but does not necessarily include all the active cells which are connected to it via static percolation paths. Thus the aggregates in our model are usually smaller than the clusters of static percolation. We assume that at the time step of its creation, each aggregate is given an identity (label) that remains with it until it disappears—either because it is absorbed into another aggregate or because all its cells become inactive. Thus each aggregate is represented by its label and a list of its cells.

Each time step consists of three stages: merging, randomization, and splitting. Merging characterizes the natural tendency of firms to capture new business opportunities. In practice, the process by which firms seek out new business opportunities takes time and is an ongoing process. Our focus on aggregates (as opposed to percolating clusters) is intended to capture this feature. Randomization characterizes the natural evolution of markets. Some business opportunities die out or become unprofitable while new ones emerge. The splitting stage represents the tendency of firms to divest activities when they judge that certain components of the firm would be more profitable when organized as separate firms [63].

The size $S_i(t)$ of aggregate i at time step t is determined as the number of its cells after the splitting stage. At the merging stage, we first create a randomized list of all the aggregates. Then, starting at the top of this list, each aggregate explores its neighbors for absorption and thus becomes a "predator." For each cell taken in the sequential order from the list of cells of the predator, we examine whether any of its 3^d-1 neighboring cells belongs to a different aggregate, which we immediately include in the list of "preys" aggregates. Next all the prey aggregates whose size is not larger than the current size of the predator are being absorbed by the predator one by one [64]. The cells of all the absorbed preys are included in the list of cells of the predator. The absorbed preys as well as the current predator are excluded from the randomized list of aggregates, and we select the next predator from the top of this list until the list is empty.

In the randomization stage, we apply a birth and death process to every cell of the lattice; i.e., each active cell becomes inactive with probability D and each inactive cell becomes active with probability B. A new active cell is considered an independent aggregate with a new identity. The inactive cells are excluded from the lists of cells of the existing aggregates, and the aggregates with no cells cease to exist and lose their identities.

Finally, at the splitting stage, we examine each aggregate for connectivity. If the death of an active cell causes an aggregate to become disconnected, the aggregate splits into several component aggregates. The largest of these retains the identity of the original aggregate and the rest are given new identities.

At the end of each time step, we calculate the growth rates for each aggregate i which existed at time steps t and $t-1$:

$$g_i \equiv \ln[S_i(t)/S_i(t-1)]. \tag{1}$$

We then start a new time step by returning to the merging stage. For the first $t_0=L^d$ time steps, we discard the size and growth statistics. This time is several orders of magnitude larger than the relaxation time of the system t_r, which we determine as the maximal age of a finite aggregate.

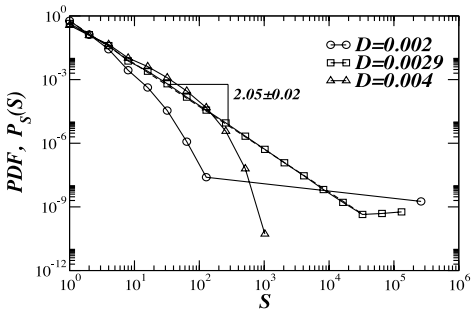

FIG. 1. The probability density function (PDF) of aggregate size S, $P_S(S)$, with the same birth probability $B=0.002$ but different death probabilities D for the lattice size $L=800$. With $D=0.0029$, the $P_S(S)$ is best fit by a power law with exponent $\tau=2.05\pm0.02$.

(The percolating aggregate which always exists above the percolation critical point never dies, and thus its age is equal to the total simulation time). We found that t_r is of the same order of magnitude as $1/D$—i.e., the average survival time of each individual active cell. In our simulation, it is unreasonable to make $D<1/L^d$ because otherwise no dynamics will be observed in one time step. Thus to be on the safe side we select $t_0=L^d$, which is always larger than t_r even for very small D. Thereafter, we analyze the statistics of the sizes and growth rates of all recorded aggregates. We have tested whether t_0 time steps are sufficient for the system to reach its steady state, after which the properties of the system remain stable. In order to do this we fit the averages of a quantity of interest as a function of time as $a-b/t^\alpha$, where a, b, and α are parameters of the fit, and check that $b/t_0^\alpha<\epsilon$, where ϵ is the error bar of this quantity averaged for $t_0<t<10t_0$.

For simplicity, we study 2D lattices, but we expect that the growth rate distributions will be of the same functional form in any dimension if one replaces the 2D values of the critical exponents by the corresponding values of the critical exponents of the d-dimensional static percolation.

III. SIMULATION RESULTS

A. Distribution of aggregate sizes $P_S(S)$

We first choose a large size lattice of 800×800 to decrease the finite-size effect. In order to find the critical state, we first fix B and vary the value of D from 10^{-4} to 0.5. We next plot the probability density of the aggregates of various sizes in double-logarithmic scale and find D for which the graph has the best straight line fit. We have applied this criterion since it is known that, at the critical point of static percolation, the cluster size distribution follows the power law $P_S(S)\sim S^{-\tau}$ up to the largest possible cluster, which can occupy up to L^{d_f} cells of the lattice, where $d_f=91/48=1.89$ and $\tau=1+d/d_f=187/91=2.05$ for $d=2$. For lattice size of $L=800$, the largest cluster can thus occupy up to 50% of all the active lattice cells. In fitting the graph, we also disregard the aggregates of sizes 1 and 2. As Fig. 1 shows, the best fit for $B=0.002$ is achieved when D is around 0.0029. We find that the exponent τ is around 2.05 ± 0.02 which coincides with the value for the static percolation [65]. In the following, we will use B_o and D_o to denote the values of B and D

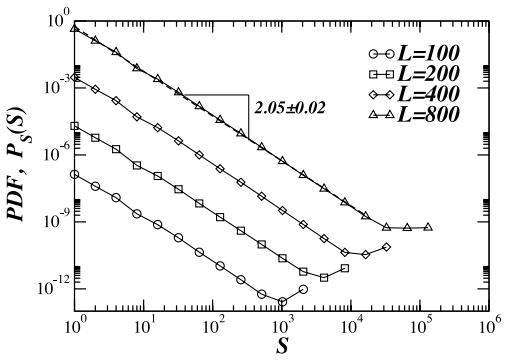

FIG. 2. The probability density function of aggregate size S, $P_S(S)$, at the critical state ($B_o=0.002$ and $D_o=0.0029$) for different lattice sizes L. The critical states with the same power-law exponents $\tau=2.05\pm0.02$ are obtained using the same B_o and D_o, which implies that the critical states are very stable. For clarity, the curves $L=100$, $L=200$, and $L=400$ are offset by e^{-15}, e^{-10}, and e^{-5}, respectively.

at the point of the best power-law fit which we postulate to coincide with the critical point.

If D is smaller than D_o, the system is above the percolation threshold, which implies that there are some giant aggregates which do not break apart in splitting process. While if D is larger than D_o, the system is below the percolation threshold and the distribution $P_S(S)$ acquires the exponential cutoff, with no aggregates above a certain size.

Next we confirm the existence of a critical state by testing the value of τ for different lattice sizes. As shown in Fig. 2, for different lattice sizes ($L=100$, 200, 400, and 800), we obtain the same value of τ: $\tau=2.05\pm0.02$.

The pair of values (B_o,D_o) determines the critical line in the phase diagram of the system. From Fig. 3, we see that D_o increases as B_o increases. When B_o is smaller than 0.05, the relationship between B_o and D_o is nearly linear, but when B_o is larger than 0.1, the curve acquires a significant negative curvature and practically levels at $D_o=0.42$ for $B_o\approx1$.

The phase diagram obtained can be well explained since the size distribution of the aggregates is measured after the

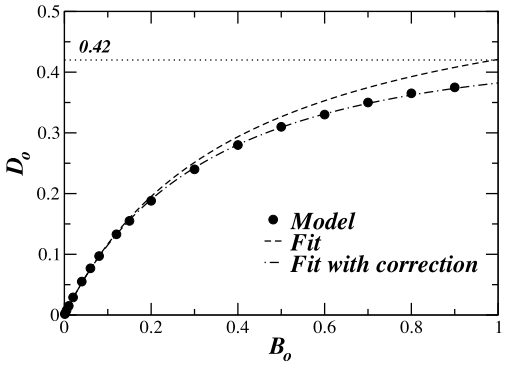

FIG. 3. The critical line (the relation between D_o and B_o) on the phase diagram of the model. The upper part of the curve means that the system is below percolation threshold, and lower part suggests that the system is above percolation threshold. The curve (for small B_o region) can be well fit by Eq. (2). When using Eq. (2) where the P_c is replaced by $P_c+\alpha B_o$ ($\alpha=0.04$), the whole curve is well fit.

splitting stage of the time step, at which the number of cells in the merged aggregates is decreased by factor of $1-D_o$ and the new active cells which appear at the randomization stage have not yet participated in merging. Thus the effective fraction of the cells which are responsible for the connectivity of large clusters is $(1-D_o)B_o/(B_o+D_o)$. Since we are at the critical point, this effective fraction should coincide with the critical probability P_c for static percolation. This condition yields

$$D_o \approx \frac{B_o(1-P_c)}{B_o+P_c}. \tag{2}$$

As Fig. 3 shows, the above expression fits well the phase diagram of the model for small B_o. Here $P_c=1-0.593=0.407$ which is the cell percolation threshold for square lattices with eight neighbors [66].

However, the argument above is not appropriate to explain the curve for large B_o. The reason is the difference between the speeds of the merging and splitting process. For large D_o, the aggregate immediately splits into many small clusters, some of which may not be neighbors, while during each merging step an aggregate can merge only with its neighbors. Thus the splitting cannot be repaired in the next merging stage and the value of D_o computed using Eq. (2) overestimates the actual value of D_o corresponding to the critical state.

This effect can be taken into account by assuming that at criticality the fraction of the cells responsible for connectivity must be larger than P_c by a small correction term αB_o, where α is a proportionality coefficient which can be found by fitting the simulation results. Fig. 3 also shows the fitting curve with $\alpha=0.04$.

To avoid the difficulties in the analytical treatment of the model, we also study a simplified version of the model in which aggregates exactly coincide with the clusters of the static percolation at any given moment of time. In this model, each elementary time step consists of moving a randomly chosen active cell from one place of the lattice to another and determining whether such a move results in splitting of the cluster to which this active cell has belonged before the move and merging of the clusters adjacent to the new location of this active cell. In this model, P_c coincides with static percolation concentration of the active cells, but the properties of distribution of the aggregate growth rates coincide with those discussed in the following section for our original model.

B. Distribution of aggregate growth rates $P_g(g)$

Figure 4 shows the probability density function (PDF) $P_g(g|S)$ of the growth rates g for aggregates of sizes $2^{k-1} \leqslant S < 2^k$ for $k=1,2,3,\ldots$, $L=800$, $B_o=0.002$, and $D_o=0.0029$. We see that, for the aggregates of medium sizes, $P_g(g|S)$ has a central peak and pronounced wings. For small aggregates, such as size 1 or 2, we cannot see the left part of the PDF since the minimum value of S is 1. In the most extreme case, one aggregate with size 1 can grow 256 times in one merging cycle. This happens if it has eight neighboring aggregates of size 1, 2, 4,…, 128, and they are selected

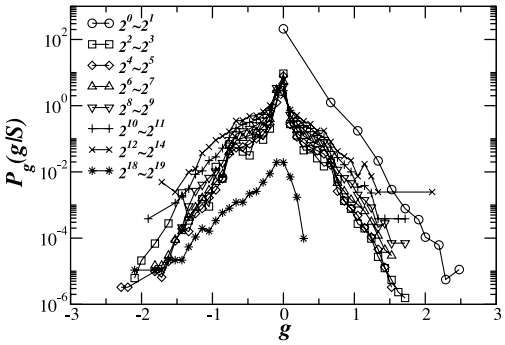

FIG. 4. The probability density function of aggregate growth rates, $P_g(g)$, for different aggregate sizes S at the critical state ($B_o=0.002$ and $D_o=0.0029$) for the lattice $L=800$. The shapes of $P_g(g)$ for intermediate S are very similar with central peak and broad wings. For clarity, the $P_g(g)$ for $S=2^0-2^1$ is offset by a factor of e^3 and the one for $S=2^{18}-2^{19}$ by a factor of e^{-5}.

for merging in the ascending order of their size. Thus $g_{\max}=\ln(256/1)\approx 5.54$. In contrast, the $P_g(g|S)$ for large S nearly misses the right part because due to finite-size effects, large aggregates cannot grow larger than the entire lattice. Thus the average growth rate $\bar{g}(S)$ is positive for small S, is close to zero for intermediate S, and is negative for large S (Fig. 5).

Figure 4 also suggests that at the critical state, the standard deviation $\sigma(S)$ of the distribution $P_g(g|S)$ increases with S. In other words, the larger aggregates are more volatile and less stable than the small aggregates. This observation is confirmed by Fig. 6. Moreover, we see that for large systems at the critical state,

$$\sigma(S) \sim S^{\beta}, \tag{3}$$

with $\beta \approx 0.20$.

Figure 7 shows the growth rate distributions scaled by $\sigma(S)$. One can see that the scaled distributions collapse on a single master curve: $P_g(g|S)=P_g(g/\sigma(S))/\sigma(S)$. This observation suggests that at the critical state the growth process is universal for aggregates of different sizes.

We can explain Eq. (3) using the concept of "red" bonds, which is frequently used in percolation theory [67]. The red

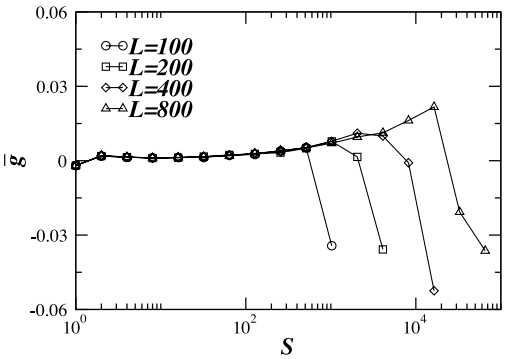

FIG. 5. The relation between mean growth rate of aggregates $\bar{g}$ and aggregate size S at the critical state ($B_o=0.002$ and $D_o=0.0029$) on different lattice sizes L. As it shows, $\bar{g}$ is approximately independent of S at the critical state.

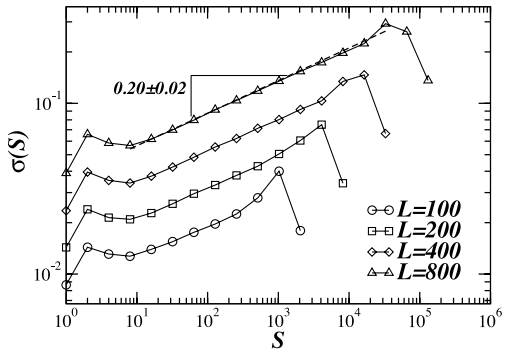

FIG. 6. The relationship between standard deviation of aggregate growth rates, $\sigma(S)$, and aggregate size, S, best fit by a power law $\sigma(S) \sim S^{\beta}$ for different lattice sizes L at the same critical state ($B_o=0.002$ and $D_o=0.0029$). Each curve gives $\beta \approx 0.20$. For clarity, the curves $L=100$, $L=200$, and $L=400$ are offset by $e^{-1.5}$, $e^{-1.0}$, and $e^{-0.5}$, respectively.

bonds are defined as such that their removal destroys the connectivity of a cluster. It is known that the fractal dimension of the red bonds $d_{\mathrm{red}}=1/\nu$, where ν is the correlation length exponent, $\nu=4/3$ for $d=2$ [68]. If D is small, the probability of the aggregate splitting is proportional to the number of the red bonds in this aggregate. Typically the aggregate splits into two approximately equal parts. Analogously, the probability to merge is proportional to the number of red bonds in the aggregate perimeter, which at the percolation threshold is proportional to the mass of the aggregate itself. The linear size of the aggregate, $R \sim S^{1/d_f}$, where d_f is the fractal dimension of the percolation cluster, $d_f=91/48$ for $d=2$ [69,70]. Thus the number of the red bonds in the aggregate and its perimeter scales as $R^{d_{\mathrm{red}}} \sim S^{d_{\mathrm{red}}/d_f}$. Accordingly, the probability p of the cluster to split and to merge in one cycle scales as $p \sim S^{d_{\mathrm{red}}/d_f}$. The probability that the size of the aggregate remains roughly unchanged in one cycle of the model is thus $1-2p$. Let us assume that in the event of a merger or splitting the aggregate size changes by factor of 2 (Fig. 8). Indeed, this assumption is corroborated by the shape of $P_g(g|S)$ which has a sharp peak near

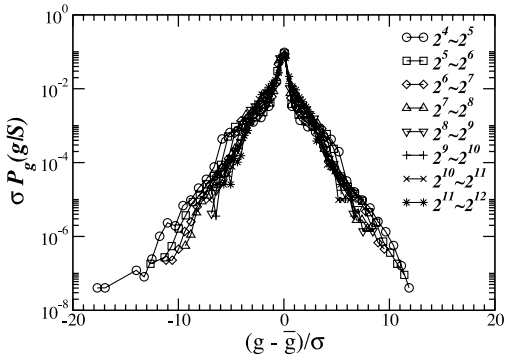

FIG. 7. The rescaled probability density function of aggregate growth rates, $\sigma(S)P_g(g|S)$, for different aggregate sizes S at the critical state ($B_o=0.002$ and $D_o=0.0029$) for the lattice $L=800$. For intermediate aggregate sizes, the $P_g(g|S)$ are collapsed onto a single curve, which implies that the $P_g(g|S)$ may follow the same functional form.

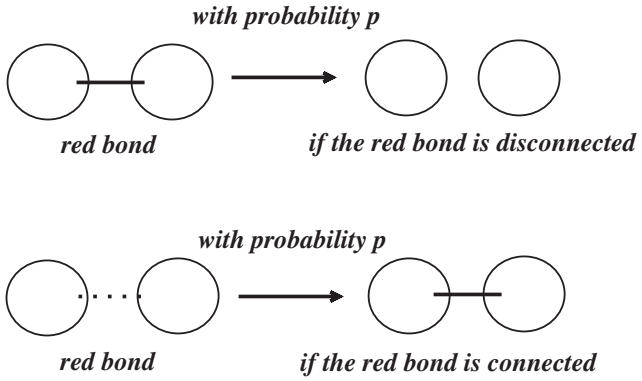

FIG. 8. Schematic explanation of $\beta \approx 0.20$ for small B_o by a typical growth pattern for a sample aggregate. We assume that each circle is of the same size. If there is a red bond connecting two components of an aggregate with probability p, the red bond is disconnected and the aggregate decreases its size by factor of 2. If a disconnected red bond is connected between two separated aggregates with probability p, the aggregate increases its size by factor of 2.

$g=0$ and two symmetric wings which abruptly drop near $g \approx \pm 0.7 \approx \pm \ln 2$ (Fig. 4). If our assumption is valid, then the variance of the growth rates can be estimated as

$$\sigma^2 = p(\ln 2)^2 + (1-2p)0 + p\left(\ln \frac{1}{2}\right)^2,$$

$$= 2p(\ln 2)^2.$$

Accordingly, $\sigma \sim \sqrt{p} \sim S^{d_{\mathrm{red}}/2d_f}$. Thus $\beta = d_{\mathrm{red}}/2d_f$. In $d=2$, we have $\beta = 18/91 \approx 0.198$, in excellent agreement with our simulation results. In higher dimensions which may be more appropriate for modeling the economy, we can predict the value of β using the known values of d_{red} and d_f for static percolation. Specifically for $d \geqslant 6$ (which is the upper critical dimension for percolation) we have $d_{\mathrm{red}}=1$ and $d_f=4$. Thus, for $d \geqslant 6$, one can expect $\beta = 1/8$.

Note that $\beta \approx 0.20$ is only valid for small B_o because the simple probabilistic arguments presented above are valid only if the average number of red bonds created or destroyed per cluster in one cycle of the model is relatively small. The mapping of our model onto a static percolation is also valid only for small B_o. Indeed, Fig. 9 shows a pronounced decrease of β for increasing B_o.

IV. DISCUSSION AND CONCLUSION

We explored the statistical properties of aggregate growth rate in a percolation model. Our principal findings are as follows: (i) there exists a critical state in our model for which the PDF of aggregate size is a power law, (ii) the distribution of aggregate growth rates at the critical state has a central peak with broad wings and the mean growth rate of aggregates is practically independent of size, (iii) the standard deviation of aggregate growth rates is power-law related, with a positive exponent, to the aggregate size, and (iv) the distribution of growth rates conditioned on aggregate size col-

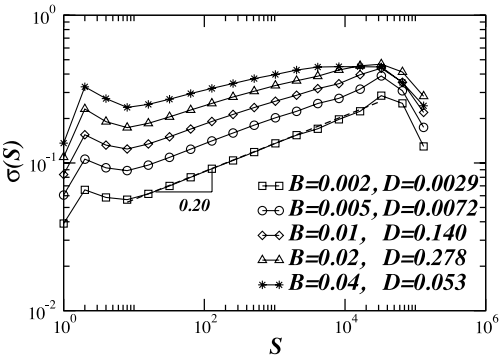

FIG. 9. The power-law relationship between the standard deviation of aggregate growth rates, $\sigma(S)$, and aggregate size S for the lattice size $L=800$ at different critical states. Note that the value of β decreases as B_o increases, which means that $\beta \approx 0.20$ is valid only for small B_o.

lapses onto a single curve, which is consistent with the possibility that a universal functional form exists to express the PDF of aggregate growth rate at the critical state.

The model represents a preliminary attempt to incorporate principles of economics into models of critical phenomena in order to explain the statistical properties of the growth of business firms. We have demonstrated that the growth rates of aggregates—the quantities in the model that are natural to interpret as business firms—have some but not all of the key features observed in the data on firm growth. At this stage of the research, the differences between the model and the data do not cast doubt on the validity of the approach because there are other additional economic factors that can be incorporated into the model. In particular, the economics literature stresses that there is a cost to coordinating additional activities within a single firm and that different activities should not coexist within a single firm if the gains from coordination do not outweigh the costs. In our model, adjacent active cells always merge, which implicitly embodies the assumption that the gains from coordination always outweigh the costs. By moving from a two-state to a three-state Potts model, it should be possible to make assumptions that more readily reflect economic reality.

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