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Robustness of n interdependent networks with partial support-dependence relationship

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Abstract – We study both analytically and numerically the robustness of n interdependent networks with partial support-dependence relationship, which reflects real-world networks more realistically. For a starlike network of n Erdős-Rényi (ER) networks, we find that the system undergoes from second-order to first-order phase transition as coupling strength q increases. Moreover, we notice that the region of the first-order transition becomes larger, while the region of the second-order transition becomes smaller as the number of networks n increases. However, for a starlike network of n scale-free (SF) networks, the system undergoes from second-order through hybrid-order to first-order phase transition as q increases. Furthermore, we also observe that the region of the first-order transition remains constant and appears only for $q = 1$, however, the region of hybrid-order transition gradually becomes larger and the region of the second-order transition becomes smaller as n increases. For a looplike network of n ER networks, we find the giant component p_∞ to be independent of the number of networks. Additionally, when the average degree of networks increases, the region of the first-order transition becomes smaller and the region of the second-order transition becomes larger. For the case of n ER networks with partial support-dependence relationship, as average supported degree $\tilde{k} \to \infty$, n coupled networks become independent and only second-order transition is observed, which is similar to $q = 0$.

Introduction. – In the past years, the study of interdependent networks attracted more and more attention [1–16]. Recently, Buldyrev *et al.* [1] introduced a dependency model with bidirectional links that defines one-to-one correspondence between nodes of two networks. Surprisingly, a broader degree distribution increases the vulnerability of interdependent networks to random failure, which is opposite to how a single network behaves. Two important generalizations have been proposed from this model: i) Parshani *et al.* [2] discussed the case of two partially interdependent networks. By analyzing this model, their finding shows that reducing the coupling between the networks leads to a change from first-order transition to second-order transition at a critical point. ii) Gao *et al.* [3] focused on studying the case of n interdependent networks. Their result suggests that the classical percolation theory extensively studied in physics and mathematics is a limited case of a general theory of percolation in n interdependent networks with $n = 1$. These studies on interdependent networks assume a one-to-one correspondence dependency condition between the nodes of any two networks. However, in the real world, interdependency between two infrastructure networks is usually not of this type. Quite recently, Shao *et al.* [4] introduced a model with multiple support-dependence between all nodes of two networks. They studied cascading failures in two fully coupled networks, where multiple support-dependence relations are randomly built between nodes of two networks. However, in many real systems, more than two networks depend on each other. And, when examining the features of real networks, we also observe that not all nodes between any two networks of n networks have a support-dependence relationship. Here, based on these motivations, we generalize Shao *et al.* model [4] by analyzing the robustness of n interdependent networks with partial support-dependence relationship under random attack, which can model real networks better.

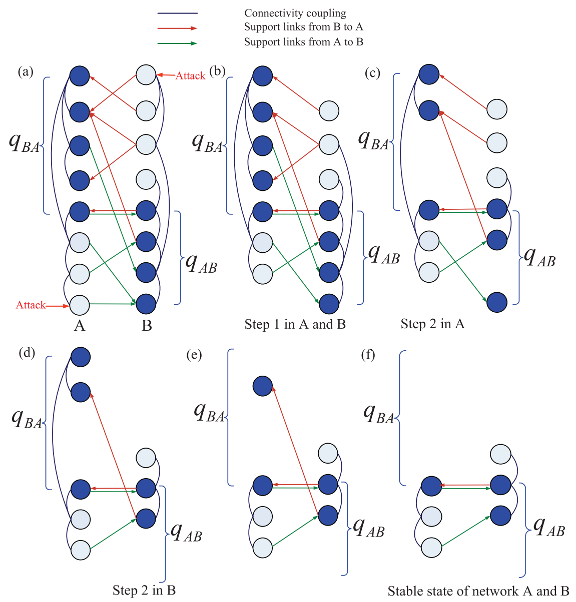

Fig. 1: (Colour on-line) Demonstration of cascading failures for two networks A and B with support-dependence relationship. Without loss of generality, we assume that the sizes of networks A and B are $N_A = N_B = 8$. And, the fraction of dependent nodes within networks A and B are $q_{BA} = \frac{5}{8}$ and $q_{AB} = \frac{1}{2}$, which are represented by blue dots, while the other white dots represent nondependent nodes. The blue curves represent connectivity links within the network, whereas directed arrows represent support links connecting support nodes in one network to dependent nodes in the other network. Correspondingly, red arrows are from B to A and green arrows are from A to B.

The model. – For each network of n networks, there exists two types of nodes: dependent nodes and nondependent nodes. Dependent nodes in one network might be supported by nodes of other networks. On the contrary, nondependent nodes do not need nodes from other networks to support them. Furthermore, a functional node of dependent nodes within one network should satisfy both of the following conditions: i) to have at least one functional support node in other networks and ii) to belong to the giant component of functional nodes in the network it belongs to [4]. A functional node of nondependent nodes within one network just needs to satisfy condition ii). For the case of any two networks A and B with number of nodes N_A and N_B of n networks, nodes are connected by connectivity links within each network, with degree distributions $P_A(k)$ and $P_B(k)$. We assume that nodes of network B support a fraction q_{BA} of nodes in network A, which are dependent nodes within network A. And, nodes of network A support a fraction q_{AB} of nodes in network B, which are dependent nodes within network B, as shown in fig. 1(a). The support-dependence relationship is randomly built between dependent nodes of A (or B) and all of the nodes of network B (or A). In addition, support links, which connect support nodes in one network to dependent nodes in the other network, are represented by unidirectional arrows. The support degree $\tilde{k}_A$ (or $\tilde{k}_B$) of a node in network A (or B) denotes that the node is supported by $\tilde{k}_A$ (or $\tilde{k}_B$) nodes in network B (or A), where $\tilde{k}_A$ (or $\tilde{k}_B$) satisfies the support degree distribution $\tilde{P}^A(\tilde{k}_A)$ (or $\tilde{P}^B(\tilde{k}_B)$).

For the process of cascading failures, initially, both networks are attacked and a fraction $1-p_A$ and $1-p_B$ of nodes in network A and B, are randomly removed, respectively. As shown in fig. 1(b), at step 1, the connectivity and dependency links of the attacked nodes are removed in both networks. When treating nodes in network A at step t, we assume that all their support nodes in network B, which are found to be functional at step $t-1$, are still functional [4]. At step 2 of network A, according to condition i), the nodes in network A, which do not receive any support from remaining nodes of network B at step 1 are removed. Then, according to condition ii), the nodes which do not belong to the giant component of network A are also removed, as shown in fig. 1(c). All the failed nodes of network A will lead to failures of support links starting from them. Similarly, when treating nodes in network B at step t, we assume that all their support nodes in network A, which are found to be functional at the current step t, are still functional [4]. Therefore, nodes in network B, which neither receive any support from the remaining functional nodes of network A nor belong to giant component of network B, are also removed at step 2, as shown in fig. 1(d). This process of cascading failures will continue until no further nodes and links removal occurs, as shown in fig. 1(f).

Theoretical framework. – In this section, we will demonstrate the theoretical framework for cascading failures of n networks. For network i, we assume that there are l neighbor networks $j_1, \cdots, j_m, \cdots, j_l$ supporting it. Without loss of generality, we study cascading failures of network i and one of its neighbor networks, j_m. For one node in network i, there are $\tilde{k}_{j_m i}$ support nodes randomly chosen from network j_m, the probability of having no functional support nodes in network j_m at step t is

$$\beta_t^{j_m i} = q_{j_m i} \sum_{\widetilde{k}_{j_m i}=0}^{\infty} \widetilde{P}^{j_m i}(\widetilde{k}_{j_m i})(1-p_{t-1}^{(j_m)})^{\widetilde{k}_{j_m i}} = q_{j_m i} \widetilde{G}^{j_m i}(1-p_{t-1}^{(j_m)}), \quad (1)$$

where a fraction of nodes $q_{j_m i}$ in network i directly depend on nodes of network j_m, a fraction of nodes $p_{t-1}^{(j_m)}$ in network j_m are functional nodes of network j_m at step $t-1$, and $\widetilde{G}^{j_m i}$ is the generating function of the support degree distribution $\widetilde{P}^{j_m i}(\widetilde{k}_{j_m i})$. Therefore, the remaining fraction of nodes in network i at step t is

$$x_t^{(i)} = p_i(1-\beta_t^{j_m i}), \quad (2)$$

where p_i denotes the remaining fraction of nodes in network i, after initially removing a fraction of $1-p_i$ nodes. By analyzing cascading failures between network

i and all of its neighbor networks, when $N_i \to \infty$, the fraction of remaining functional nodes in network i is

$$x_t^{(i)} = p_i \prod_{m=1}^{l} [1 - q_{j_m i} \widetilde{G}_{j_m i}(1 - p_{t-1}^{(j_m)})], \qquad m = l. \tag{3}$$

For the neighbor network j_m, we assume that there are $r+1$ neighbor networks $i, s_1, \cdots, s_h, \cdots, s_r$ supporting it. Therefore, we randomly choose $\widetilde{k}_{ij_m}$ support nodes in network i, and the probability that a node in network j_m has no support nodes in network i at step t is

$$\begin{aligned}\beta_t^{(ij_m)} &= q_{ij_m} \sum_{\widetilde{k}_{ij_m}=0}^{\infty} \widetilde{P}^{ij_m}(\widetilde{k}_{ij_m})(1 - p_t^{(i)})^{\widetilde{k}_{ij_m}} \\ &= q_{ij_m} \widetilde{G}^{ij_m}(1 - p_t^{(i)}). \end{aligned} \tag{4}$$

From the above analysis, the probability that a node in network j_m has no functional support nodes in network s_h $(h = 1, \cdots, r)$ at step t is

$$\begin{aligned}\beta_t^{s_h j_m} &= q_{s_h j_m} \sum_{\widetilde{k}_{s_h j_m}=0}^{\infty} \widetilde{P}^{s_h j_m}(\widetilde{k}_{s_h j_m})(1 - p_{t-1}^{(s_h)})^{\widetilde{k}_{s_h j_m}} \\ &= q_{s_h j_m} \widetilde{G}^{s_h j_m}(1 - p_{t-1}^{(s_h)}). \end{aligned} \tag{5}$$

Therefore, for $N_i \to \infty$, the fraction of remaining functional nodes in network j_m at step t is

$$\begin{aligned} x_t^{(j_m)} &= p_{j_m}(1 - q_{ij_m} \widetilde{G}^{ij_m}(1 - p_t^{(i)})) \\ &\quad \cdot \prod_{h=1}^{l} [1 - q_{s_h j_m} \widetilde{G}_{s_h j_m}(1 - p_{t-1}^{(j_m)})]. \end{aligned} \tag{6}$$

Then, we analyze cascading failures within networks by applying condition ii). The generating function of the degree distribution $P^i(k)$ of network i $(i = 1, 2, \ldots, n)$ is [4,7]

$$G_{i0} = \sum_{k=0}^{\infty} P^i(k) x^k. \tag{7}$$

The generating function of the underlying branching process is

$$G_{i1} = \frac{G'_{i0}(x)}{G'_{i0}(1)}. \tag{8}$$

After removing a fraction of $1 - p_i$ nodes from network i, new generating functions of the degree distribution and of the underlying branching process are

$$\begin{cases} G_{i0}(x, p_i) = G_{i0}(1 - p_i(1 - x)), \\ G_{i1}(x, p_i) = G_{i1}(1 - p_i(1 - x)). \end{cases} \tag{9}$$

At step t, the fraction of nodes which belong to the giant component of remaining nodes in network i, $x_t^{(i)}$, is

$$g^{(i)}(x_t^{(i)}) = 1 - G_{i0}(f_t^{(i)}, x_t^{(i)}), \tag{10}$$

where $f_t^{(i)}$ satisfies the transcendental equation

$$f_t^{(i)} = G_{i1}(f_t^{(i)}, x_t^{(i)}). \tag{11}$$

Thus, the fraction of nodes in the giant component of network i is

$$p_t^{(i)} = x_t^{(i)} g^{(i)}(x_t^{(i)}). \tag{12}$$

From eqs. (3), (6) and (12), as $t \to \infty$, $x_t^{(i)}$ and $p_{t-1}^{(i)}$ both reach constant values, $x_\infty^{(i)}$ and $p_\infty^{(i)}$. Thus, the expressions of $x_\infty^{(i)}$ and $p_\infty^{(i)}$ are

$$\begin{cases} x_\infty^{(i)} = p_i \prod_{j=1}^{l} [1 - q_{ji} \widetilde{G}_{ji}(1 - p_\infty^{(j)})], \\ p_\infty^{(i)} = x_\infty^{(i)} g^i(x_\infty^{(i)}). \end{cases} \tag{13}$$

Especially, as $l = 1$ and $q_{ji} = 1$, we observe that eq. (13) is consistent with Shao's results [4].

Robustness of a starlike network of n networks. – For n ER networks with partial support-dependence relationship, the generating functions of the degree distribution and of the underlying branching process of network i $(i = 1, 2, \ldots, n)$ are [7,10,17–25]

$$G_{i0} = G_{i1} = e^{\langle k_i \rangle (x-1)}, i = 1, 2, \ldots, n. \tag{14}$$

From eqs. (10), (11) and (14), we obtain the following equations:

$$\begin{cases} g^{(1)}(x_\infty^{(1)}) = 1 - f_\infty^{(1)} = 1 - e^{-\langle k_1 \rangle p_\infty^{(1)}}, \\ g^{(j)}(x_\infty^{(j)}) = 1 - f_\infty^{(j)} = 1 - e^{-\langle k_j \rangle p_\infty^{(j)}}, \end{cases} \tag{15}$$

where $p_\infty^{(1)}$ and $p_\infty^{(j)}$ denote the fractions of the giant component of the central network and surrounding networks j $(j = 2, \ldots, n)$ at the end of the cascading process. In this letter, we choose the support degree distributions $\widetilde{P}^{j1}(\langle \widetilde{k}_{j1} \rangle)$ and $\widetilde{P}^{1j}(\langle \widetilde{k}_{1j} \rangle)$ to be Poisson distributions. Then, we get

$$\begin{cases} q_{j1} \widetilde{G}_{j1}(1 - p_\infty^{(1)}) = q_{j1} e^{-\langle \widetilde{k}_{j1} \rangle p_\infty^{(1)}}, \\ q_{1j} \widetilde{G}_{1j}(1 - p_\infty^{(j)}) = q_{1j} e^{-\langle \widetilde{k}_{1j} \rangle p_\infty^{(j)}}. \end{cases} \tag{16}$$

And, eq. (13) becomes

$$\begin{cases} x_\infty^{(1)} = p_1 \prod_{j=1}^{n} [1 - q_{j1} e^{-\langle \widetilde{k}_{j1} \rangle p_\infty^{(j)}}], \\ p_\infty^{(1)} = p_1 \prod_{j=1}^{n} [1 - q_{j1} e^{-\langle \widetilde{k}_{j1} \rangle p_\infty^{(j)}}](1 - e^{-\langle k_1 \rangle p_\infty^{(1)}}), \\ x_\infty^{(j)} = p_j [1 - q_{1j} e^{-\langle \widetilde{k}_{1j} \rangle p_\infty^{(1)}}], \\ p_\infty^{(j)} = p_j [1 - q_2 e^{-\langle \widetilde{k}_2 \rangle p_\infty^{(1)}}](1 - e^{-\langle k_j \rangle p_\infty^{(j)}}). \end{cases} \tag{17}$$

Let $p_j = p_2$, $q_{j1} = q_1$, $q_{1j} = q_2$, $\langle k_1 \rangle = \langle k_j \rangle = \langle k \rangle$, $\langle \widetilde{k}_{j1} \rangle = \langle \widetilde{k}_1 \rangle$ and $\langle \widetilde{k}_{1j} \rangle = \langle \widetilde{k}_2 \rangle$, from eqs. (15), (16) and (17), we get

$x_\infty^{(j)} g^{(j)}(x_\infty^{(j)}) = x_\infty^{(2)} g^{(2)}(x_\infty^{(2)})$, $p_\infty^{(j)} = p_\infty^{(2)}$ and $f_\infty^{(j)} = f_\infty^{(2)} (j = 2, 3, \dots, n)$. Then eqs. (11) and (17) can be transformed to

$$\begin{cases} x_\infty^{(1)} = p_1[1 - q_1 e^{-\langle \widetilde{k}_1 \rangle p_\infty^{(2)}}]^{n-1}, \\ p_\infty^{(1)} = p_1[1 - q_1 e^{-\langle \widetilde{k}_1 \rangle p_\infty^{(2)}}]^{n-1}(1 - e^{-\langle k \rangle p_\infty^{(1)}}), \\ f_\infty^{(i)} = e^{-\langle k \rangle p_\infty^{(1)}}, \\ f_\infty^{(2)} = e^{-\langle k \rangle p_\infty^{(2)}}, \\ x_\infty^{(2)} = p_2[1 - q_2 e^{-\langle \widetilde{k}_2 \rangle p_\infty^{(1)}}], \\ p_\infty^{(2)} = p_2[1 - q_2 e^{-\langle \widetilde{k}_2 \rangle p_\infty^{(1)}}](1 - e^{-\langle k \rangle p_\infty^{(2)}}). \end{cases} \tag{18}$$

and

$$\begin{cases} p_\infty^{(1)} = -\dfrac{\ln f_\infty^{(1)}}{\langle k \rangle}, \\ p_\infty^{(2)} = -\dfrac{\ln f_\infty^{(2)}}{\langle k \rangle}. \end{cases} \tag{19}$$

From eqs. (18) and (19), we have

$$\begin{cases} p_\infty^{(2)} = -\dfrac{\ln\left\{\frac{1}{q_1}\left[1 - \left(\frac{p_\infty^{(1)}}{p(1-e^{-\langle k \rangle p_\infty^{(1)}})}\right)^{\frac{1}{n-1}}\right]\right\}}{\langle \widetilde{k}_1 \rangle}, \\ p_\infty^{(1)} = -\dfrac{\ln\left\{\frac{1}{q_2}\left[1 - \frac{p_\infty^{(2)}}{p(1-e^{-\langle k \rangle p_\infty^{(2)}})}\right]\right\}}{\langle \widetilde{k}_2 \rangle}. \end{cases} \tag{20}$$

From eqs. (19) and (20), $f_\infty^{(1)}$ and $f_\infty^{(2)}$ can be solved

$$f_\infty^{(2)} = \left\{\frac{1}{q_1}\left[1 - \left(\frac{-\ln f_\infty^{(1)}}{p\langle k \rangle(1 - f_\infty^{(1)})}\right)^{\frac{1}{n-1}}\right]\right\}^{\frac{\langle k \rangle}{\langle \widetilde{k}_1 \rangle}}, \tag{21}$$

$$f_\infty^{(1)} \neq 1; \ \forall f_\infty^{(2)}, \ f_\infty^{(1)} = 1,$$

$$f_\infty^{(1)} = \left\{\frac{1}{q_2}\left[1 + \frac{\ln f_\infty^{(2)}}{p\langle k \rangle(1 - f_\infty^{(2)})}\right]\right\}^{\frac{\langle k \rangle}{\langle \widetilde{k}_2 \rangle}}, \tag{22}$$

$$f_\infty^{(2)} \neq 1; \ \forall f_\infty^{(1)}, \ f_\infty^{(2)} = 1.$$

We verify our theory, eq. (20), by comparing theoretical predictions with simulation results for different coupling strength q, as shown in fig. 2(a). Additionally, from fig. 2(a), we observe that the giant component of the central network undergoes from second- to first-order phase transition as q increases. Furthermore, by analyzing the graphical solution of eqs. (21) and (22), the critical fraction p_c can be solved by finding the touching point of curves $f_\infty^{(2)}(f_\infty^{(1)})$ and $f_\infty^{(1)}(f_\infty^{(2)})$. Thus, the critical fraction p_c can be obtained from the tangential condition $\frac{\mathrm{d}f_\infty^{(2)}}{\mathrm{d}f_\infty^{(1)}} \frac{\mathrm{d}f_\infty^{(1)}}{\mathrm{d}f_\infty^{(2)}} = 1$. Therefore, we get the first-order transition point p_c^I for strong-coupling strength $q \geqslant q_c$ and the second-order transition point p_c^{II} for weak-coupling strength $q < q_c$ for different n. Moreover, we notice that as the number of networks n increases, the critical threshold q_c decreases. This means that the region of the first-order transition becomes larger, while the region of the second-order transition becomes smaller as n increases. For average supported degree $\langle \widetilde{k} \rangle \to \infty$, we see that the expressions of $p_\infty^{(1)}$ and $p_\infty^{(2)}$ of eq. (18) are consistent with the single ER network's expression, which means that in a central network only second-order phase transition occurs, like in a single ER network. In fact, from fig. 2(c), one can also see that p_c approaches the percolation threshold $\frac{1}{\langle k \rangle}$ of the single ER network, which is the same as $q = 0$.

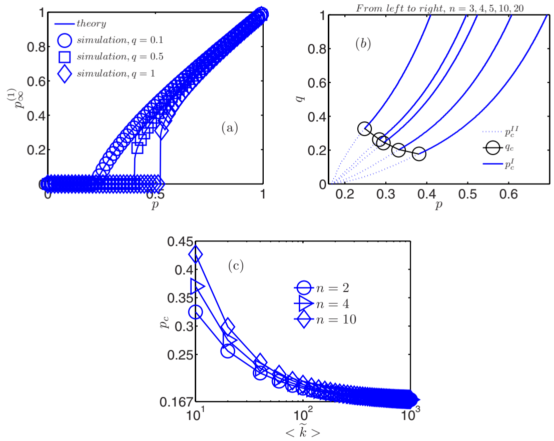

Fig. 2: (Colour on-line) (a) For a starlike network of n ER networks, comparison between simulations and theory of $p_\infty^{(1)}$ as a function of p for different q, with parameters $\langle \widetilde{k} \rangle = 6$, $\langle k \rangle = 6$, and $n = 5$. In simulation, $N_i = N = 10^5$ and the results are averaged over 50 realizations. (b) Coupling strength q as a function of p for different n with parameters $\langle \widetilde{k} \rangle = 6$, $\langle k \rangle = 6$. (c) The critical fraction p_c as a function of $\langle \widetilde{k} \rangle$ for different n with parameters $\langle k \rangle = 6$ and $q = 1$.

For SF networks, the degree distribution is $P(k) = ck^{-\lambda}, m < k < M$, where λ is the width of the distribution, and k, M, m are the degree, maximum degree, minimum degree, respectively. The generating functions of the degree distribution and the underlying branching process of network i are [26]

$$\begin{cases} G_{i0}(x) = \sum_m^M [(\frac{m}{k})^{\lambda-1} - (\frac{m}{k+1})^{\lambda-1}]x^k, \\ G_{i1}(x) = \frac{\sum_m^M [(\frac{m}{k})^{\lambda-1} - (\frac{m}{k+1})^{\lambda-1}]kx^{k-1}}{\sum_m^M [(\frac{m}{k})^{\lambda-1} - (\frac{m}{k+1})^{\lambda-1}]k}. \end{cases} \tag{23}$$

To simplify the theoretical framework of SF networks, we choose the parameters $p_j = p_2$, $q_{j1} = q_1$, $q_{1j} = q_2$, $\lambda_1 = \lambda_j = \lambda$, $\langle \widetilde{k}_{j1} \rangle = \langle \widetilde{k}_1 \rangle$, $\langle \widetilde{k}_{1j} \rangle = \langle \widetilde{k}_2 \rangle$ $(j = 2, 3, \dots, n)$. Thus, from eqs. (10), (11), (14) and (16), we get $x_\infty^{(j)} g^{(j)}(x_\infty^{(j)}) = x_\infty^{(2)} g^{(2)}(x_\infty^{(2)})$, $p_\infty^{(j)} = p_\infty^{(2)}$ and $f_\infty^{(j)} = f_\infty^{(2)}$, as follows:

$$\begin{cases} f_\infty^{(1)} = \frac{\sum_m^M [(\frac{m}{k})^{\lambda-1} - (\frac{m}{k+1})^{\lambda-1}]k(1 - x_\infty^{(1)} + x_\infty^{(1)} f_\infty^{(1)})^{k-1}}{\sum_m^M [(\frac{m}{k})^{\lambda-1} - (\frac{m}{k+1})^{\lambda-1}]k}, \\ f_\infty^{(2)} = \frac{\sum_m^M [(\frac{m}{k})^{\lambda-1} - (\frac{m}{k+1})^{\lambda-1}]k(1 - x_\infty^{(2)} + x_\infty^{(2)} f_\infty^{(2)})^{k-1}}{\sum_m^M [(\frac{m}{k})^{\lambda-1} - (\frac{m}{k+1})^{\lambda-1}]k}; \end{cases} \tag{24}$$

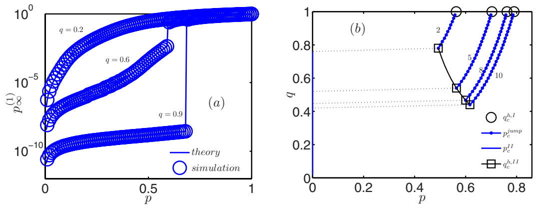

Fig. 3: (Colour on-line) (a) For a starlike network of n SF networks, comparison between simulations and theory for $p_\infty^{(1)}$ for different q, with parameters $n=5$, $\langle\widetilde{k}\rangle=5$, $\lambda=2.7$, $N_i=10^5$, $i=1,2,3,4,5$. The simulation results are averaged over 50 realizations. (b) q as a function of p for $n=2,5,8,10$ for a starlike network of n SF networks, other parameters are the same as (a). In this numerical solution, we choose NOI $=t$ and $\xi=10^{-30}$ for $p_t^{(1)}-p_{t+1}^{(1)}<\xi$.

$$\begin{cases} x_\infty^{(1)}=p_1[1-q_1e^{-\langle\widetilde{k}_1\rangle p_\infty^{(2)}}]^{n-1},\\ x_\infty^{(2)}=p_2[1-q_2e^{-\langle\widetilde{k}_2\rangle p_\infty^{(1)}}],\\ p_\infty^{(1)}=x_\infty^{(1)}(1-f_\infty^{(1)}),\\ p_\infty^{(2)}=x_\infty^{(2)}(1-f_\infty^{(2)}). \end{cases} \tag{25}$$

Here, we compare simulations with theory from eqs. (24) and (25) for a starlike network of five SF networks as shown in fig. 3(a). From fig. 3(a), we observe that the central network undergoes from second-order through hybrid-order to first-order phase transition as the coupling strength q increases. For hybrid-order transition, $p_\infty^{(1)}$ jumps at p_c^{jump} from a large value to a small value and then continuously decreases to zero as p decreases. We get p_c^{jump} from NOI, which is easily identified by the sharp peak characterizing the first-order and hybrid-order transition point [2,26]. Furthermore, the threshold $q_c^{h,II}$ where second-order transition turns into hybrid-order transition can be easily identified as shown in fig. 3(b) [26]. We also observe that the critical threshold $q_c^{h,I}$, where hybrid-order transition turns into first-order transition, keeps constant at $q=1$ and $q_c^{h,II}$ gradually decreases as n increases. Therefore, as n increases, the region of first-order transition only occurs at $q=1$, the region of hybrid transition becomes larger and the region of second-order transition becomes smaller as n increases.

Robustness of a looplike network of n networks. – In this section, we study the robustness of a looplike network of n ER networks with partial support-dependence relationship. When a fraction of $1-p_i$ $(i=1,2,\ldots,n)$ nodes are removed from network i and $t\to\infty$, f_i, x_i and p_i keep constant and equal to f_∞, x_∞ and p_∞ by setting parameters $p_i=p$, $q_{l(l+1)}=q_{(l+1)l}=q_{1n}=q_{n1}=q$, $\langle\widetilde{k}_{l(l+1)}\rangle=\langle\widetilde{k}_{(l+1)l}\rangle=\langle\widetilde{k}_{1n}\rangle=\langle\widetilde{k}_{n1}\rangle=\langle\widetilde{k}\rangle$ and $\langle\widetilde{k}_i\rangle=\langle\widetilde{k}\rangle$ $(i=1,2,\ldots,n)$. Then, by eqs. (10), (11), (12), (13), (14) and (15), we get

$$\begin{cases} x_\infty=p(1-qe^{-\langle\widetilde{k}\rangle p_\infty})^2,\\ f_\infty=e^{-\langle\widetilde{k}\rangle p_\infty}. \end{cases} \tag{26}$$

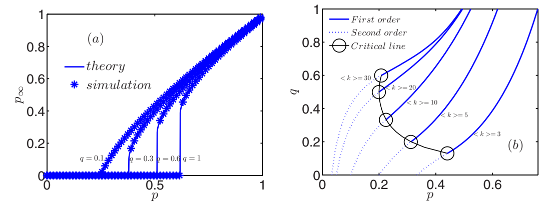

Fig. 4: (Colour on-line) (a) For a looplike network of n ER networks, comparison of theoretical results and simulations for p_∞ for different q, with parameters $n=5$, $\langle\widetilde{k}_i\rangle=5$, $\langle k\rangle=5$ and $N_i=10^5, i=1,2,3,4,5$. (b) The coupling strength q as a function of p for different $\langle k\rangle$ with $\langle\widetilde{k}\rangle=5$.

$$p_\infty=p(1-e^{-\langle k\rangle p_\infty})(1-qe^{-\langle\widetilde{k}\rangle p_\infty})^2. \tag{27}$$

From eqs. (26) and (27), we see that the giant component of a looplike network of n ER networks is independent of n. This is different from a starlike network of n ER networks. In addition, from fig. 4(a), we see that the network undergoes from second-order to first-order phase transition as the coupling strength q increases. From eqs. (26) and (27), we also obtain

$$p_\infty=\frac{-\ln f_\infty}{\langle k\rangle}. \tag{28}$$

From eqs. (27) and (28), we get

$$f_\infty=e^{-p\langle k\rangle(1-f_\infty)\left(1-qf_\infty^{\frac{\langle\widetilde{k}\rangle}{\langle k\rangle}}\right)^2}. \tag{29}$$

From the above analysis, we obtain the critical fraction p_c^I for the first-order transition:

$$p_c^I=\frac{1-f_\infty+f_\infty\ln f_\infty}{2q\langle\widetilde{k}\rangle f_\infty^{\frac{\langle\widetilde{k}\rangle}{\langle k\rangle}}(1-f_\infty)^2\left(1-qf_\infty^{\frac{\langle\widetilde{k}\rangle}{\langle k\rangle}}\right)}. \tag{30}$$

By solving eq. (29) for $f_\infty\to1$, we get the critical fraction p_c^{II} for the second-order transition:

$$p_c^{II}=\frac{1}{\langle\widetilde{k}\rangle(1-q)^2}. \tag{31}$$

Applying both eqs. (30) and (31), we get the critical threshold

$$\begin{cases} p_c=\dfrac{(\langle k\rangle+4\langle\widetilde{k}\rangle)^2}{16\langle k\rangle\langle\widetilde{k}\rangle^2},\\ q_c=\dfrac{\langle k\rangle}{\langle k\rangle+4\langle\widetilde{k}\rangle}. \end{cases} \tag{32}$$

From fig. 4(b), we observe that the critical threshold q_c separating first-order and second-order transition increases as the average degree $\langle k\rangle$ increases, *i.e.*, as $\langle k\rangle$ increases, the region of the first-order transition becomes smaller and the region of the second-order transition becomes larger. Furthermore, from eq. (27), we observe that as $\langle\widetilde{k}\rangle\to\infty$, the expression of the giant component of the looplike network of n networks changes into the expression of the single ER network, which is consistent with the above analysis.

Summary. – For n networks with partial support-dependence relationship, we analyze the robustness of two cases, a starlike network of n ER, SF networks and a looplike network of n ER networks. For the case of a starlike network of n ER networks, the region of the first-order transition becomes larger, while the region of the second-order transition becomes smaller as the number of networks n increases. For the case of a starlike network of n SF networks, the region of the first-order transition remains constant and is only at $q = 1$, however, the region of the hybrid-order transition gradually becomes larger and the region of the second-order transition becomes smaller as n increases. For the case of a looplike network of n ER networks, we find that p_∞ is independent of n. Moreover, as $\langle k \rangle$ increases, the region of the first-order transition becomes smaller and the region of the second-order transition becomes larger.

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