Universal behavior of cascading failures in interdependent networks

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Catastrophic and major disasters in real-world systems, such as blackouts in power grids or global failures in critical infrastructures, are often triggered by minor events which originate a cascading failure in interdependent graphs. We present here a self-consistent theory enabling the systematic analysis of cascading failures in such networks and encompassing a broad range of dynamical systems, from epidemic spreading, to birth–death processes, to biochemical and regulatory dynamics. We offer testable predictions on breakdown scenarios, and, in particular, we unveil the conditions under which the percolation transition is of the first-order or the second-order type, as well as prove that accounting for dynamics in the nodes always accelerates the cascading process. Besides applying directly to relevant real-world situations, our results give practical hints on how to engineer more robust networked systems.

cascading failure | interdependent network | spreading | robustness

In recent years, lots of studies concentrated on catastrophic events affecting the Internet, power grids, or other critical infrastructures (1–6). Most such major disasters are in fact triggered by minor events, which may originate a cascading failure in interdependent graphs (7–12). Understanding the robustness of these networks with respect to minor perturbations is of utmost importance for preventing system crashes (13, 14). An example is power and communication networks: Power stations administer energy to communication nodes and depend on them at the same time for control, so that malfunctions may spread from one network to the other (15). Cascading failures have been studied intensively in statistical physics on single graphs or on networks that interact with (and/or depend on) each other (16, 17).

To capture the spreading mechanisms in such graphs, one needs a mathematical scaffold able to seize 2 basic aspects: the structural and functional interdependence between the networks, and the spreading of failures within each single graph. So far, the first issue was dealt with by examining changes in the structure connectivity caused by node dependence between networks (12, 15, 18), and the critical properties of the failure process were unveiled with the help of percolation theory (19–23). Interdependency between network constituents fundamentally alter the percolation properties (8, 24–30). In particular, Parshani et al. (22) found that, when a critical fraction of nodes in one network fails at a weak (strong) interdependence level between the networks, the system undergoes a second-order (first-order) phase transition due to recursive processes of cascading failures. As for the second issue, studies concentrated on the conditions and outcome for cascading failures (31–34). Different from structural failures caused by removed nodes, overload failures usually propagate in the system through invisible paths (35–40). In particular, Barzel and Barabási (39) developed a self-consistent theory able to analyze the case of a perturbation localized in one node which spreads within the structure of the graph.

In this paper, we contribute a fresh mathematical framework for cascading failures, providing answers to fundamental questions such as: 1) What happens when a *finite* fraction of nodes experiences a dynamical overload on a single network? 2) How can the failure process caused by dynamical perturbations and interdependence be properly characterized? 3) Which factors determine the cascade to be a second- or a first-order phase transition? 4) What are the *universal properties* that are common to the spreading of cascading failures for different dynamical systems, dependence strengths, and network structures?

Results

Cascade Size in Single Networks. We start by considering the case of a fraction of nodes being perturbed in a graph. Here, each node $i\,(i = 1, \ldots, N)$ of a pristine network is assigned a time-dependent variable $x_i(t)$, obeying a generic rate equation (39)

$$\frac{dx_i}{dt} = W\left(x_i\left(t\right)\right) + \sum_{j=1}^{N} A_{ij}\, Q\left(x_i\left(t\right), x_j\left(t\right)\right). \qquad \textbf{[1]}$$

Significance

Catastrophic events affecting technological or critical infrastructures are often originated by a cascading failure triggered by marginal perturbations, which are on their turn localized in one of the many interdependent graphs describing the systems. Understanding the robustness of these graphs is therefore of utmost importance for preventing crashes and/or for engineering more efficient and stalwart networked systems. Here we give a fresh framework by means of which cascading failures can be described in a very rich variety of dynamical models and/or topological network structures and which provides a series of quantitative answers able to predict the extent of the system's failure.

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Table 1. Dynamical model and failure size in ER networks

Dynamics	Model	$\omega(1-p)$	$\langle C\rangle$
$\mathcal{B}$	$\frac{dx_i}{dt}=F-Bx_i-\sum_{j=1}^{N}A_{ij}\tilde{R}x_ix_j$	$1-p\sum_0^\infty P_C(C)p^C$	$\sim\delta^{-1}$
$\mathcal{BD}$	$\frac{dx_i}{dt}=-Bx_i^\kappa+\sum_{j=1}^{N}A_{ij}x_j^\rho$	$1-pe^{(p-1)\langle C\rangle}p^{\langle C\rangle-\alpha\delta^{\frac{2}{3}}<C^{\frac{2}{3}}>}$	$\alpha^{\frac{-3}{2}}\delta^{-1}<k^{\frac{3}{2}}>$
$\mathcal{E}$	$\frac{dx_i}{dt}=-Bx_i+\sum_{j=1}^{N}A_{ij}R(1-x_i)x_j$	$1-pe^{(p-1)\langle C\rangle}p^{\langle C\rangle-\alpha\delta^{\frac{1}{2}}\langle C\rangle}$	$\alpha^{-1}\delta^{\frac{-1}{2}}<k>$
$\mathcal{R}$	$\frac{dx_i}{dt}=-Bx_i+\sum_{j=1}^{N}RH(x_j)$	$1-pe^{(p-1)\langle C\rangle}p^{\langle C\rangle-\alpha\delta\langle C^2\rangle}$	$\alpha^{\frac{-1}{2}}\delta^{\frac{-1}{2}}<k^{\frac{1}{2}}>$

$W(x_i)$ accounts for the evolution of x_i in the absence of network interactions, A_{ij} is the graph adjacency matrix, and $Q(x_i,x_j)$ is a function describing all pairwise interactions. With appropriate $W(x_i)$ and $Q(x_i,x_j)$, Eq. **1** can be mapped exactly into several dynamical models (Table 1 and *SI Appendix*), such as epidemic processes (where x_i represents the local probability of infection) (41, 42), biochemical dynamics (in which x_i represents the concentration of a reactant) (43), birth–death processes (in which x_i represents the population of a given species at site i) (44), and regulatory dynamics (in which x_i is the expression level of a gene) (45).

In the absence of perturbations, each node i of a pristine network reaches its asymptotic steady state $\tilde{x}_i$. A tolerance coefficient δ is introduced, and when perturbations are present, the node i readjusts its asymptotic state into x_i. If $\mid 1-x_i/\tilde{x}_i\mid>\delta$, the node i is considered to fail (i.e., it is removed from the network together with all its incident links), and the value x_i is permanently set to zero. Therefore, a perturbation affecting a given node j (consisting of a permanent vanishing of $\tilde{x}_j$) generates a readjustment of the network, which may lead to the failure of other nodes, triggering this way a cascade which ends only when all of the nodes in the final graph have values within the fixed network's tolerance.

In ref. 39, it was shown that the cascade size C_i (the number of nodes failing after a perturbation affecting node i) is $C_i\sim k_i\left(\delta k_i^{1-\gamma}\right)^{-\frac{1}{\beta+1}}$, where k_i is the degree of the node i, and γ and β are parameters accounting, respectively, for the local impact and the propagation dynamics of the perturbation. Thus, one can get the distribution of the failure size caused by perturbing one node. When $\beta+\gamma\neq 0$, the connectivity of one node comes out to be $k_i=\alpha(\delta^{\frac{1}{\beta+1}}C_i)^{\frac{\beta+1}{\beta+\gamma}}$; here, α is a mapping coefficient between the real cascading failure size and the node degree. Notice that for biochemical dynamics with $\beta=0$ and $\gamma=0$, one gets $\langle C\rangle=C_i\sim\delta^{-1}$. Furthermore, the correspondence between the node connectivity and the cascading failure size is one to one (k_i to C_i). Hence, we can get the cascading failure size distribution of each component from the degree distribution within an arbitrary network. Assuming that the graph's degree distribution is $P_k(k)$, then the cascading failure size distribution of the system (see the derivation in *SI Appendix*) follows

$$P_C(C)=P_k(\alpha\delta^{\frac{1}{\beta+\gamma}}C^{\frac{\beta+1}{\beta+\gamma}}). \quad [2]$$

Let us now discuss the case of a fraction $1-p$ of nodes being perturbed randomly. If the failure range of each node only covers its nearest neighbors, the failure size is $\varsigma(1-p)=1-p\sum_{k=0}^{\infty}P_k(k)p^k$, where the overlap conditions are considered as well. And applying it to the cascading cases, the failure size $\omega(1-p)$ is

$$\omega(1-p)=1-p\sum_{C=0}^{\infty}P_C(C)p^C. \quad [3]$$

Table 1 illustrates 4 distinct dynamical models and their corresponding failure size [according to Eq. **3**] on single Erdös-Rényi (ER) networks. The results are in good agreement with simulations, as shown in Fig. 1. Notice that, for $\mathcal{BD}$, $\mathcal{E}$, and $\mathcal{R}$ models in ER networks, we can approximate the failure size (see *SI Appendix* for details) as $\omega(1-p)\approx 1-pe^{(p-1)\langle C\rangle}p^{<C>-\alpha<C^{\frac{\beta+1}{\beta+\gamma}}>\delta^{\frac{1}{\beta+\gamma}}}$.

Interdependent Graphs. Let us now move from isolated to interdependent networks (46–50), and let us consider 2 networks, A and B (having, respectively, N_A and N_B nodes [with, in general, $N_A\leq N_B$]) with a fraction q_A (q_B) of network A's nodes (of network B's nodes) depending on nodes in network B (A). If node i in network A (B) stops functioning, the dependent node j in network B (A) fails to work as well. The nodes in network A or B have dynamical activity. In the absence of perturbations, each node i of networks A and B reaches its asymptotic steady state $\tilde{x}_i$. Once again, a tolerance coefficient δ is introduced, and if due to perturbations, the new state x_i of node i verifies $\mid 1-x_i/\tilde{x}_i\mid>\delta$, then the node is considered to be nonfunctional, its value x_i is permanently set to zero, and it is removed from the network together with all its incident links. In this way, when a perturbation is added to a fraction of

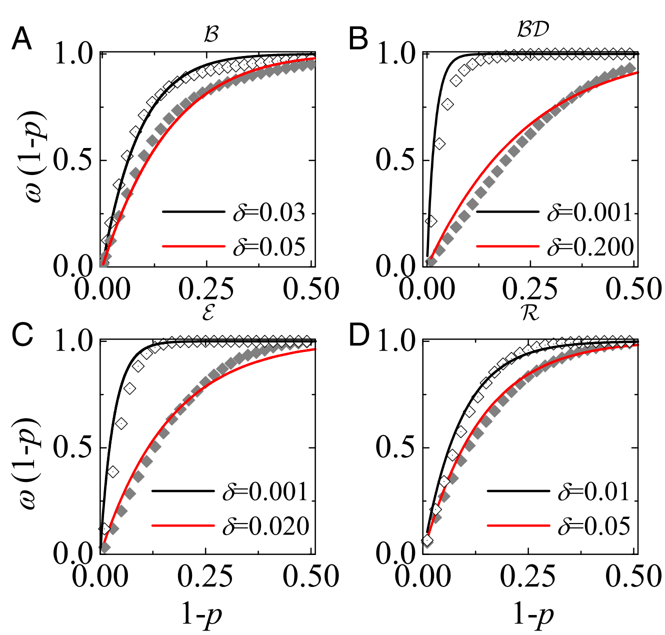

Fig. 1. The cascade size $\omega(1-p)$ vs. the fraction $1-p$ of perturbed nodes for the 4 models of Table 1. All data refer to ensemble averages over 100 different realizations of ER networks with N =1,000. Diamonds and solid lines are used for indicating the simulation results and the analytical solutions in Table 1, respectively. (*A*) Comparison of simulation results and theoretical solutions of $\omega(1-p)$ of ER networks with $\langle k\rangle$ = 5 for $\mathcal{B}$. (*B–D*) Same as *A*, but the results for $\mathcal{BD}$ with $\langle k\rangle$ = 2 (*B*), for $\mathcal{E}$ with $\langle k\rangle$ = 5 (*C*), and for $\mathcal{R}$ with $\langle k\rangle$ = 5 (*D*) are presented, respectively.

nodes in network A randomly, cascading failures in the system may occur. The cascading failure process is schematically illustrated in Fig. 2: The failure caused by the fraction $1-p$ of perturbed nodes diffuses initially within a single graph, then it crosses into the dependent network, and an iterative process starts which ends when there are no new failures or when the system collapses completely. Notice that this interdependency scheme differs from (and is complementary of) a recent study (51), where, instead, dependency was ruled by the local value of an order parameter describing the transition to a collective state.

Failures with the $\mathcal{E}$, $\mathcal{B}$, $\mathcal{BD}$, and $\mathcal{R}$ Dynamics. We now specialize to the case in which the interacting graphs A and B are hosting the dynamical models of Table 1. The nodes of network $A(B)$ are connected with degree distribution $P_A(k)$ $[P_B(k)]$, and a fraction q_A (q_B) of nodes in network $A(B)$ depends on nodes in network $B(A)$ with no-feedback condition (15).

When an infinitesimally small fraction $1-p$ of nodes in network A is perturbed, the fraction of intact nodes is $\lambda_1' = p' = 1-\omega(1-p)$ (at the A's perturbed steady state), and the fraction of nodes which belongs to the giant component of A is $\lambda_1 = \lambda_1' g_A(\lambda'_1)$, g_A being the fraction of nodes belonging to the giant components of network A. As the functioning of a fraction (q_B) nodes in B depends on that of the linked nodes in A, the fraction of failing nodes in B is $(1-\lambda_1)q_B = [1-\lambda_1' g_A(\lambda'_1)]q_B$. From Eq. **3**, the remaining fraction of nodes in B is $\varphi_1' = 1-\omega\left((1-\lambda'_1 g_A(\lambda'_1))q_B\right)$. Hence, its giant components are $\varphi_1 = \varphi_1' g_B(\varphi'_1)$, with g_B being the fraction of nodes belonging to the giant components of network B. Iteration of the above steps gives the fraction of intact nodes in A and B at each stage of the cascade failure. The general form is given by (15, 22)

$$
\begin{aligned}
&\lambda'_1 = p', \quad \lambda_1 = \lambda'_1 g_A(\lambda'_1),\\
&\varphi'_1 = 1-\omega\left(\left(1-p' g_A(\lambda'_1)\right)q_B\right), \quad \varphi_1 = \varphi'_1 g_B(\varphi'_1),\\
&\lambda_2' = p'(1-\omega((1-g_B(\varphi_1'))q_A)), \quad \lambda_2 = \lambda'_2 g_A(\lambda'_2),\\
&\vdots\\
&\varphi'_t = 1-\omega\left(\left(1-p' g_A(\lambda'_t)\right)q_B\right), \quad \varphi_t = \varphi'_t g_B(\varphi'_t),\\
&\lambda'_t = p'\left[1-\omega\left(\left(1-g_B(\varphi'_{t-1})\right)q_A\right)\right], \quad \lambda_t = \lambda'_t g_A(\lambda'_t).
\end{aligned}
\qquad [4]
$$

At the end of the cascade, no further failures occur, and the system attains its steady state. Namely, at the limit of $t\to\infty$, $\lambda_\infty{}' = \lambda'_{t+1} = \lambda'_t$, and $\varphi_\infty{}' = \varphi'_{t+1} = \varphi'_t$. Therefore, the giant components of each network are $P_{\infty,A} = \lambda_\infty = \lambda_\infty{}' g_A(\lambda_\infty{}')$ and $P_{\infty,B} = \varphi_\infty = \varphi_\infty{}' g_B(\varphi_\infty{}')$.

As compared with conclusions in refs. 15 and 22, what is striking here is the *universality* of the failure condition formula (our results in the case of no dynamical behavior are in full agreement with those of ref. 22; Fig. 3*A*), which entitles us to explore a very rich variety of dynamical models, networks' topologies, and interdependence strengths (results are reported in Fig. 3 *B–F*, as well as in *SI Appendix*).

First- and Second-Order Transitions. According to the definition of g_A and g_B (Materials and Methods), one has $g_A(\xi) = 1-f_A$ and $g_B(\xi) = 1-f_B$. Furthermore, in accordance with Eqs. **4** and **6**, these relationships can be solved with respect to f_B (f_A) and f_A (f_B). The condition leading to a first-order phase transition (where the size of the giant component abruptly changes from a

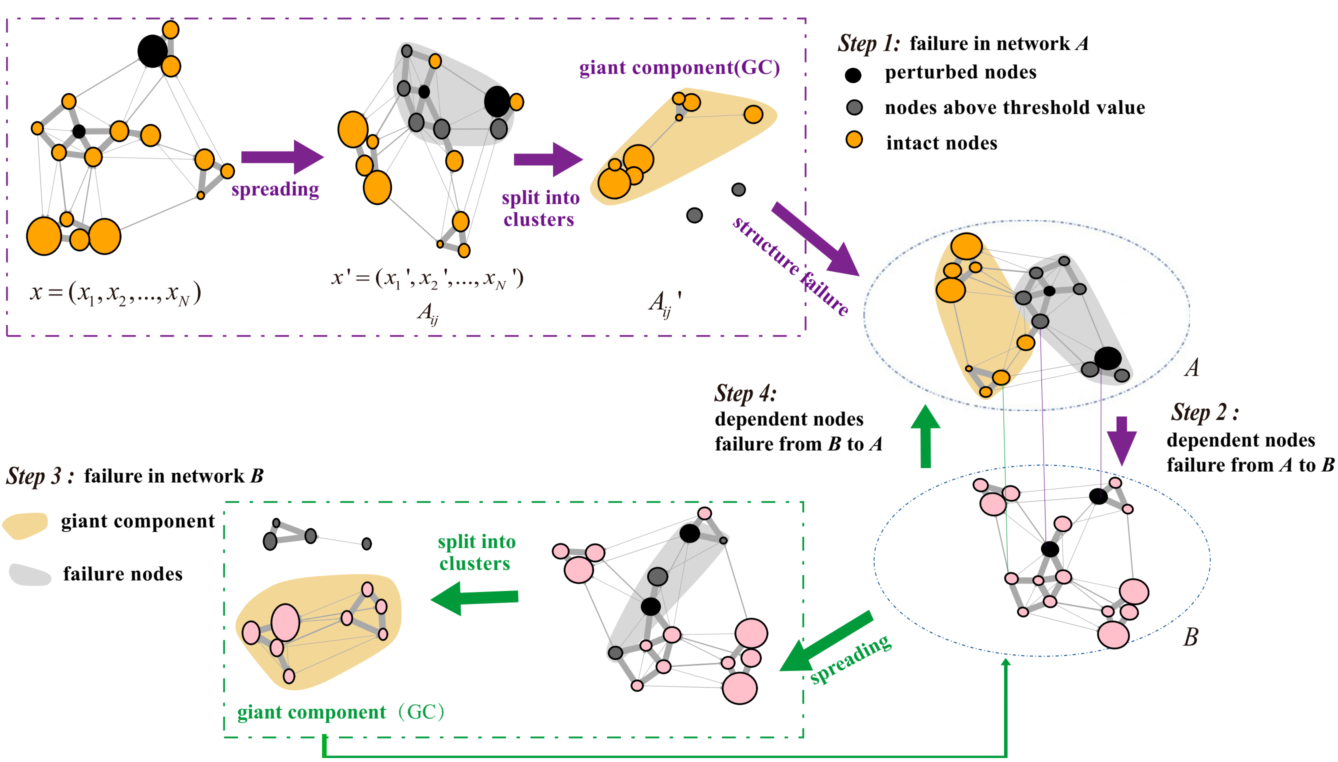

Fig. 2. Schematic illustration of the cascading failure occurring in interdependent graphs. In the absence of perturbations, each node i of networks A and B reaches its asymptotic steady state $\tilde{x}_i$. If, due now to perturbations, a given node i reaches a new state x_i such that $|1-x_i/\tilde{x}_i| > \delta$, the node is considered to be nonfunctional, its value x_i is permanently set to zero, and it is removed from the network together with all its incident links. In this way, when a permanent perturbation is added to a fraction of nodes in network A randomly, cascading failures may occur because a fraction q_A (q_B) of network A's nodes (of network B's nodes) are dependent on nodes in network B (A) in the sense that if node i in network A (B) stops functioning, the dependent node j in network B (A) fails as well.

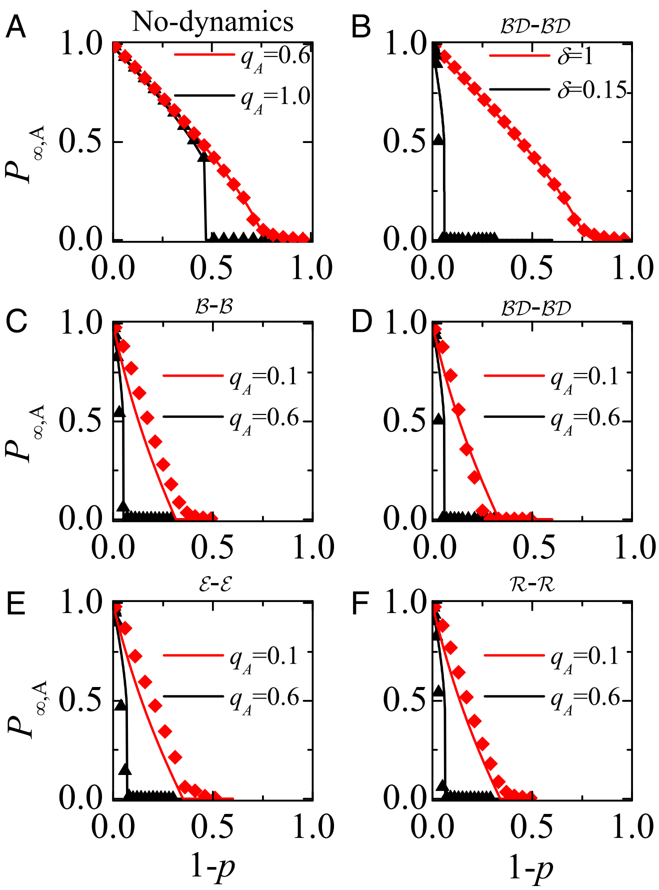

Fig. 3. $P_{\infty,A}$ (see main text for definition) vs. $1-p$, after the end of the cascade failure in ER–ER networks of size $N_A = N_B = 500$. (*A*) $P_{\infty,A}$ vs. $1-p$ in the absence of dynamical behaviors. Notice that by increasing the interdependence coupling strength, one has a change from a second- to a first-order percolation transition. (*B*) $P_{\infty,A}$ vs. $1-p$ with tolerance coefficient $\delta = 1$ and $\delta = 0.15$ for $\mathcal{BD}-\mathcal{BD}$. (*C*–*F*) Same as *A*, but now the different dynamical processes ($\mathcal{B}-\mathcal{B}$ [*C*], $\mathcal{BD}-\mathcal{BD}$ [*D*], $\mathcal{E}-\mathcal{E}$ [*E*], and $\mathcal{R}-\mathcal{R}$ [*F*]) of Table 1 are incorporated. Strikingly, the nodes' dynamics accelerates the change from second- to first-order transition. In all cases, triangles and diamonds are used for the simulations and solid lines for the analytical results.

finite value to zero) is $\frac{df_B(f_A)}{df_A}\cdot\frac{df_A(f_B)}{df_B}=1$ (22), and one can find the 3 unknowns $f_A = f_{A_1}, f_B = f_{B_1}$ and $p = p_1$ for all given values of $k_A, k_B, q_A, q_B, \delta$. The condition, instead, for the system featuring a second-order phase transition (where the size of the giant component smoothly decreases to zero) is $f_B = 1$ or $f_A = 1$, and one can find $f_A = f_{A_2}, f_B = f_{B_2}$ and $p = p_2$, which (for any fixed value of $k_A, k_B, q_A, q_B, \delta$) define a line of occurrence.

Taking the $\mathcal{BD}-\mathcal{BD}$ model in ER–ER interdependent networks as an example, Fig. 4 highlights the factors influencing the failure condition and its relationship with the percolation transition. Comparing, indeed, the case of only structural dependence (blue line in Fig. 4*A*), interdependent networks (black line) are more vulnerable, and first-order phase transitions occur more often. In Fig. 4 *B* and *D*, one notices that the critical point decreases as the network connectivity and threshold value are increased, which means that enabling the occurrence of first-order transitions here requires disturbing more nodes or increasing the number of interdependent nodes in the networks. Fig. 4*C* reports that the increase of q_B (the fraction of dependent nodes in network B) accelerates the occurrence of the first-order transition. More information on $\mathcal{B}-\mathcal{B}$, $\mathcal{E}-\mathcal{E}$, and $\mathcal{R}-\mathcal{R}$ models is reported in Fig. 5 (showing the contour maps of the transition condition) and in *SI Appendix*. The threshold value for failure spreading condition accelerates the change from the continuous failure mode (green regions) to abrupt failures (orange regions). Meanwhile, even with weak coupling strengths, the dynamic processes may lead to a first-order percolation transition.

Discussion

In this work, we have presented a generic theory of failure spreading for randomly connected graphs with arbitrary degree distributions, able to give quantitative predictions on the transition points to percolation.

Our findings show that the interdependent network within dynamical behavior always accelerates the cascading process. In other words, interdependent networks are more vulnerable, and first-order phase transitions occur more often when dynamical behaviors are considered. In particular, when the tolerance coefficient is large enough, our results are in full agreement with the case of no-dynamical-behavior. Due to its general applicability, our theory entitles us to explore a very rich variety of dynamical models and interdependence strengths. *SI Appendix* contains further analytical and numerical results on different dynamical models and different topological structures of the graphs. It is there shown that the first- and second-order phase transitions occur indeed not only in ER networks, but also for scale-free networks with the same dynamical models $\mathcal{E}$, $\mathcal{B}$, $\mathcal{BD}$, and $\mathcal{R}$. Together with contributing to a better understanding of the mechanisms underlying cascading failures in real-world interdependent networks, the importance of our results relies also on the fact that they furnish practical hints on how to engineer more robust and resilient networked systems.

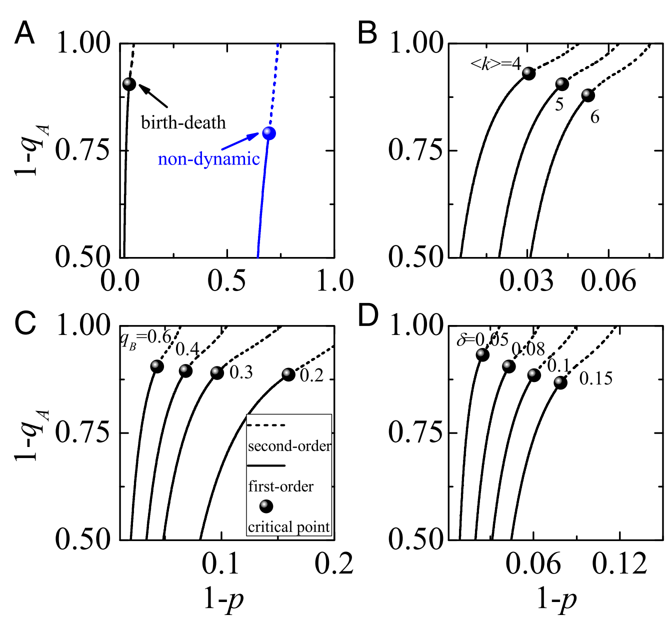

Fig. 4. (*A*) Comparison of the percolation phase transition in ER–ER networks (blue line) with the transition occurring with a $\mathcal{BD}-\mathcal{BD}$ model (black line). $1-q_A$ is the fraction of independent nodes in network *A*, and $1-p$ is the removed fraction. Solid (dashed) lines denote the first-order (second-order) percolation transition, and balls indicate the critical points. The system is more vulnerable when the dynamics is incorporated into its failure analysis. *B*–*D* show the contributions to the system failure mode of 3 factors: the average degree ($\langle k \rangle$ = 4, 5, and 6 from left to right) (*B*), the interdependence strength q_B (q_B = 0.6, 0.4, 0.3, and 0.2 from left to right) (*C*), and the tolerance coefficient (δ = 0.05, 0.08, 0.10, and 0.15 from left to right) (*D*).

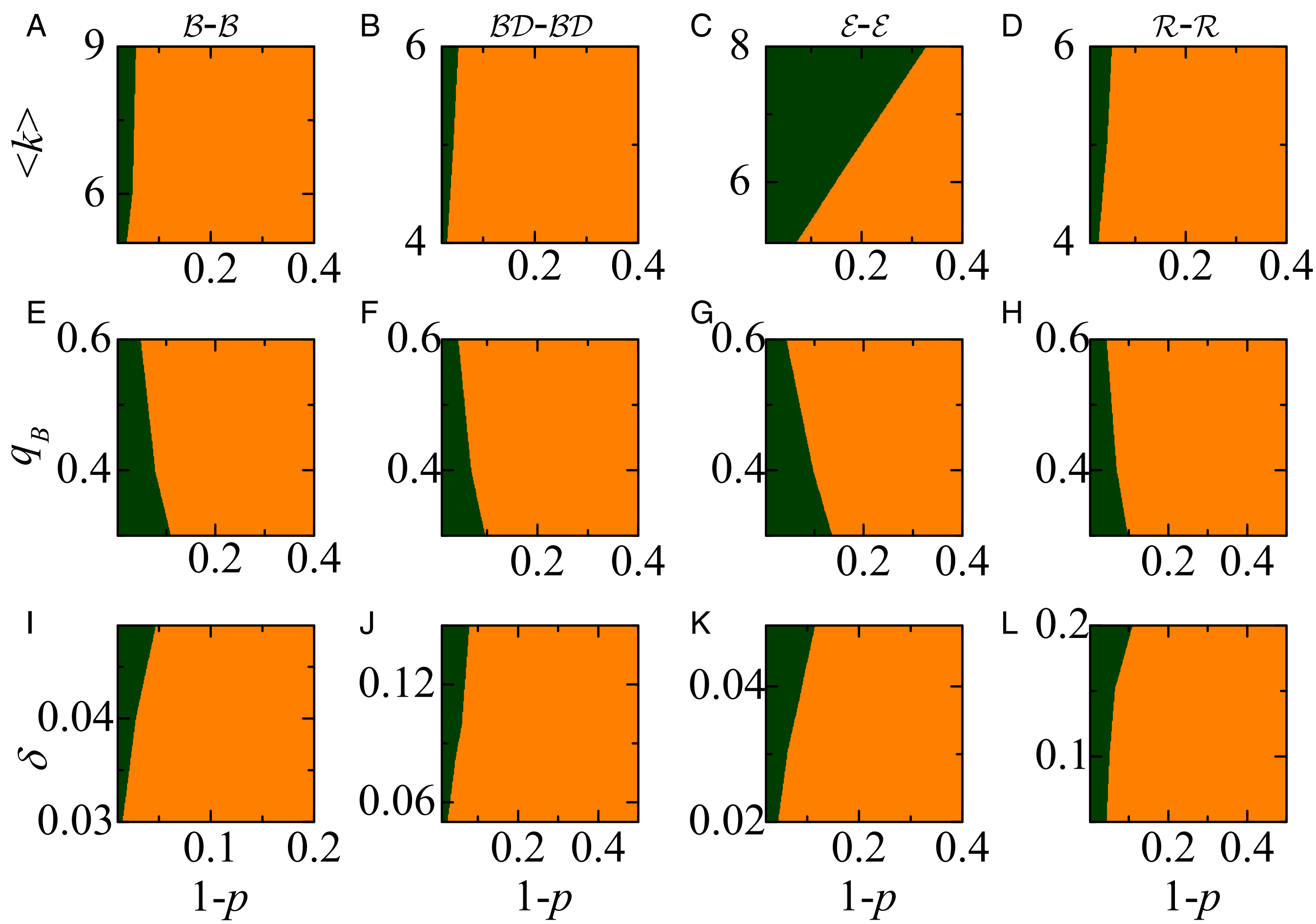

Fig. 5. Contour maps of the transition condition from continuous (green region) to abrupt failure mode (orange region) with ER–ER interdependent networks. (*A–D*) The transition condition for the models $\mathcal{B}-\mathcal{B}$ (*A*), $\mathcal{BD}-\mathcal{BD}$ (*B*), $\mathcal{E}-\mathcal{E}$ (*C*), and $\mathcal{R}-\mathcal{R}$ (*D*) when consider the impact of the average connectivity $\langle k \rangle$. (*E–H*) Same as *A–D,* but explore the impact of the coupling strength q_B for the models $\mathcal{B}-\mathcal{B}$ (*E*), $\mathcal{BD}-\mathcal{BD}$ (*F*), $\mathcal{E}-\mathcal{E}$ (*G*), and $\mathcal{R}-\mathcal{R}$ (*H*). (*I–L*) Same as *E–H* but test the contribution of the tolerance coefficient δ to the transition condition for $\mathcal{B}-\mathcal{B}$ (*I*), $\mathcal{BD}-\mathcal{BD}$ (*J*), $\mathcal{E}-\mathcal{E}$ (*K*), and $\mathcal{R}-\mathcal{R}$ (*L*), respectively.

Materials and Methods

Generating Functions. In order to estimate the giant component caused by a cascading failure, the role of the pristine graph's topological structure needs investigation. Starting from a portion $1-p$ of perturbed nodes, the fraction of failed nodes at the perturbed steady state is $1-p' = \omega(1-p)$ (according to Eq. **3**), so the random variable ξ satisfies $\xi = p'f + (1-p')$, and the giant component of the network is

$$P_\infty = p' g\left(p'\right), \qquad [5]$$

where $g\left(p'\right) = 1 - G\left(1 - p'\left(1 - f\left(p'\right)\right)\right)$, $f\left(p'\right) = H\left(p' f\left(p'\right) + 1 - p'\right)$. One can then derive the generating function of the underlying branching processes, which is $H(\xi) = G'(\xi)/G'(1)$.

Theoretical Analysis of Critical Conditions. Let us now specify our approach for the case of 2 coupled ER networks, for which we suppose a Poisson degree distribution and average degrees k_A and k_B, respectively. Then, $G_A(\xi) = H_A(\xi) = \exp\left[k_A(\xi-1)\right]$ and $G_B(\xi) = H_B(\xi) = \exp\left[k_B(\xi-1)\right]$. Eqs. **4** become $\lambda'_t = p'\left[1 - \omega\left(f_B q_A\right)\right]$, $\varphi'_t = 1 - \omega\left(\left(1 - p'\left(1 - f_A\right)\right) q_B\right)$, where f_A and f_B satisfies the transcendental equation

$$f_A = e^{\left[k_A \lambda'_t \left(f_A - 1\right)\right]}, \quad f_B = e^{\left[k_B \varphi'_t \left(f_B - 1\right)\right]}. \qquad [6]$$

At the end of the cascade process, $\lambda_t = p'\left(1-f_A\right)\left[1 - \omega\left(f_B q_A\right)\right]$ and $\varphi_t = (1-f_B)(1-\omega((1-p'(1-f_A))q_B))$. From Eq. **6**, one obtains $f_A = \exp\left[k_A p'\left[1 - \omega\left(f_B q_A\right)\right]\left(f_A - 1\right)\right]$, $f_B = \exp\left[k_B\left(1 - \omega\left(\left(1 - p'\left(1 - f_A\right)\right) q_B\right)\right)\left(f_B - 1\right)\right]$.

The first (second) of the above equations can be solved with respect to f_B (f_A), and one eventually obtains

$$\omega\left(f_B q_A\right) = 1 - \frac{\tau_A}{p'}, \qquad \omega\left(\left(1 - p'\left(1 - f_A\right)\right) q_B\right) = 1 - \tau_B, \qquad [7]$$

where $\tau_A = ln(f_A)/(k_A(f_A-1))$, $\tau_B = ln(f_B)/(k_B(f_B-1))$.

The solutions of Eqs. **7** are either $f_B(f_A)$ or $f_A = 1$ [respectively, $f_A(f_B)$ or $f_B = 1$] and are restricted to the square $0 \le f_A \le 1$, $0 \le f_B \le 1$.

$$\frac{df_B(f_A)}{df_A} \cdot \frac{df_A(f_B)}{df_B} = 1. \qquad [8]$$

When adding this condition to Eqs. **7**, one finds the 3 unknowns $f_A = f_{A_1}$, $f_B = f_{B_1}$ and $p = p_1$ for all given values of $k_A, k_B, q_A, q_B, \delta$.

The condition, instead, for the system featuring a second-order phase transition (where the size of the giant component smoothly decreases to zero) is

$$\begin{cases} 1 - \dfrac{\tau_A}{p'} - \omega\left(q_A\right) = 0, \\ 1 - \dfrac{1}{k_B} - \omega\left(\left(1 - p'\left(1 - f_A\right)\right) q_B\right) = 0. \end{cases} \qquad [9]$$

When adding $f_B = 1$ or $f_A = 1$ to Eqs. **7**, one finds $f_A = f_{A_2}$ or $f_B = f_{B_2}$ and $p = p_2$, which (for any fixed value of $k_A, k_B, q_A, q_B, \delta$) define a line of occurrence for second-order phase transitions.

So far, we have shown that, if both conditions of Eqs. **8** and **9** hold, one can obtain the critical point at which the system-failure process changes from a second- to a first-order percolation transition. Adding to Eqs. **7** the first-order condition of Eq. **8** and the second-order condition $f_B = 1$ or $f_A = 1$ allows one to find the critical parameters $f_B = f_{B_c}$, $f_A = f_{A_c}$, $p = p_c$, and $q_A = q_{A_c}$ as functions of k_A, k_B, q_B, δ.

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