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Supplementary Information for

Universal Behavior of Cascading Failures in Interdependent Networks

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This PDF file includes:

Supplementary text
Figs. S1 to S5
Table S1
References for SI reference citations

Supporting Information Text

Dynamical model. Here, we give details on the four dynamical systems that are used in the main text, namely the biochemical (B), birth-death (BD), epidemic (E), and regulatory (R) dynamics.

B: We consider protein-protein interactions (PPI), which include the process $\varphi \to X_i$ describing the synthesis of a protein i at rate F, and the process $X_i \to \varphi$ describing protein degradation at rate B. $X_i + X_j \rightleftharpoons X_i X_j$ represents the binding (unbinding) of a pair of interaction proteins at rate $R(U)$. The hetero-dimer $X_i X_j$ undergoes degradation $X_i X_j \to \varphi$ at rate Q. The dynamical equations for this system are

$$\begin{aligned} \frac{dx_i}{dt} &= F - Bx_i + \sum_{j=1}^{N} Ux_{ij} - \sum_{j=1}^{N} A_{ij} R x_i x_j, \\ \frac{dx_{ij}}{dt} &= A_{ij} R x_i x_j - (U+Q)x_{ij}, \end{aligned} \quad [1]$$

where $x_i(t)$ is the concentration of i and $x_{ij}(t)$ is the concentration of the hetero-dimer $X_i X_j$. Assuming a steady state condition for the hetero-dimer concentration, i.e. $dx_{ij}/dt = 0$, one has

$$\frac{dx_i}{dt} = F - Bx_i - \sum_{j=1}^{N} A_{ij} \tilde{R} x_i x_j, \quad [2]$$

where the effective binding rate $\tilde{R} = QR/(U+Q)$.

BD: Birth-death process has many applications in population dynamics, queuing theory, or biology. We consider a network in which the nodes represent sites, and each node i has a population x_i. Population flow is allowed between neighboring sites. This process can be described by a dynamical equation as

$$\frac{dx_i}{dt} = -Bx_i^{\kappa} + \sum_{j=1}^{N} A_{ij} x_j^{\rho}, \quad [3]$$

where the first term on the right-hand side represents the internal dynamical of site i, characterized by the exponent κ, while the second term describes the flow from i s neighboring sites j into i, which is typically linear in x_j, namely $\rho = 1$.

E: In the susceptible-infected-susceptible (SIS) model each node may be in one of two potential states: infected (I) and susceptible(S). The dynamics is given by the two processes $I + S \to 2I$, where a susceptible model is infected by one of its nearest neighbors, and $I \to S$, where an infected node is recovered, becoming susceptible again. The activity of a node, $0 \le x_i \le 1$ denotes the probability that the node is in the infected state. The dynamics of the system is governed by

$$\frac{dx_i}{dt} = -Bx_i + \sum_{j=1}^{N} A_{ij} R(1 - x_i)x_j. \quad [4]$$

R: To mimick regulatory interactions we referred to the commonly used Michaelis-Menten dynamics which take the form of

$$\frac{dx_i}{dt} = -Bx_i + \sum_{j=1}^{N} RH(x_j), \quad [5]$$

where $H(x_j)$ is the Hill function characterizing the activation/inhibition of x_i by x_j.

Network robustness with cascading failure. We begin by describing the network robustness for a single network. We assume that all N nodes in a network G be randomly assigned a degree k from a probability distribution $P_k(k)$. If we remove one node randomly, this node and its neighbors are nonfunctional. Our concern is that how many nodes are nonfunction when a fraction $1-p$ of nodes is removed randomly. In fact, this problem can be understood and mapped as the question of the node and its neighbor node being covered in a graph.

The probability that the node with degree k is not initially selected is $P_k(k)p$. We can see the probability that any one of its neighbors will not be selected is $P_k(k)p^k$, then the probability that the node in the network is not covered is $\sum_{k=0}^{\infty} P_k(k)p^{k+1}$. Hence, if a fraction $1-p$ of nodes is attacked, both the attacked node and its neighbor nodes fail (be covered) at the same time, the failure size $\varsigma(1-p)$ for the networks can be obtained as

$$\varsigma(1-p) = 1 - p\sum_{k=0}^{\infty} P_k(k)p^k. \quad [6]$$

We then discuss the network robustness with cascading failures in the presence of general dynamical systems discussed in the main text. In the absence of perturbations, each node i of a pristine network reaches its asymptotic steady state $\tilde{x}_i$. A tolerance coefficient δ is introduced, and when perturbations are present the node i readjusts its asymptotic state into x_i. If $|1 - x_i/\tilde{x}_i| > \delta$, the node i is considered to fail and the value x_i is permanently set to zero. Therefore, a perturbation affecting a given node j generates a readjustment of the network that may lead to the failure of other nodes, triggering a cascade of failures, which ends only when all the nodes in the final graph have values within the fixed network's tolerance, δ. The cascade size C_i caused by one node is $C_i \quad k_i \left(\delta k_i^{1-\gamma}\right)^{-\frac{1}{\beta+1}}$ (1), where k_i is the degree of the node i, and γ and β are parameters accounting, respectively, for the local impact and the propagation dynamics of the perturbation. Thus, when $\beta + \gamma = 0$, we obtain

$$k_i = \alpha\delta^{\frac{1}{\beta+\gamma}} C_i^{\frac{\beta+1}{\beta+\gamma}}. \tag{7}$$

Here, α is a mapping coefficient between the real cascading failure size and the node degree. We will discuss how to calculate α later on. Furthermore, the correspondence between the node connectivity and the cascading failure size is one to one (k_i to C_i). Hence, we can get the cascading failure size distribution $P_C(C)$ of each node from the degree distribution $P_k(k)$ within an arbitrary network as

$$P_C(C) = P_k(\alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}). \tag{8}$$

When examining the dynamical behavior on networks, the failure size triggered by perturbing a single node is C, the distribution of which is Eq.(8). When a fraction $(1-p)$ of nodes is perturbed, following the method of Eq.(6) (the failure nodes C can be seen as covered as well), the corresponding network failure size $\omega(1-p)$ is

$$\omega(1-p) = 1 - p\sum_{C=0}^{\infty} P_C(C)\, p^C. \tag{9}$$

Actually, Eq.(9) enables us to get the cascade failure size when a fraction $1-p$ of nodes is perturbed just with the one-node-caused failure size distribution of $P_C(C)$.

Robustness of single networks with dynamical behaviors. Here, we show details on the robustness of Erdös-Rényi (ER) networks in the presence of the four dynamical systems discussed in the main text (with parameters specified in Table S1), namely the biochemical (B), birth-death (BD), epidemic (E), and regulatory (R) dynamics. In the case of ER graphs with the degree distribution of $P_k(k) = \frac{k^{\ k}}{k!}e^{-\ k}$, from Eq.(8) one gets

$$P_C(C) = \frac{C^{\alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}}}{[\alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}]!} e^{-\ C}. \tag{10}$$

Substituting Eq. (10) into Eq. (9), one gets the cascade failure size of ER networks when a fraction $1-p$ of nodes is perturbed

$$\begin{aligned} \omega(1-p) &= 1 - p\sum_{C=0}^{\infty} \frac{C^{\alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}}}{[\alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}]!} e^{-\ C} p^C, \\ &= 1 - pe^{-\ C}\sum_{C=0}^{\infty} \frac{C^{\alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}}}{[\alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}]!} e^{-\ C} p^C. \end{aligned} \tag{11}$$

Actually, Eq.(11) enables us to get the cascade failure size when a fraction $1-p$ of nodes is perturbed just with the one-node-caused failure size distribution of $P_C(C)$.

For simplification, we denote $y = \alpha\delta^{\frac{1}{\beta+\gamma}} C^{\frac{\beta+1}{\beta+\gamma}}$, then get $C = (\frac{y}{\alpha})^{\frac{\beta+\gamma}{\beta+1}}\delta^{-\frac{1}{\beta+1}}$, so Eq.(11) becomes

$$\omega(1-p) = 1 - pe^{-\ C}\sum_{y=0}^{\infty} \frac{C^{\ y} p^y}{y!} p^{-y} p^{(\frac{y}{\alpha})^{\frac{\beta+\gamma}{\beta+1}}\delta^{\frac{-1}{\beta+1}}}. \tag{12}$$

Here, $p^{-y}p^{(\frac{y}{\alpha})^{\frac{\beta+\gamma}{\beta+1}}\delta^{\frac{-1}{\beta+1}}} = p^{C-\alpha C^{\frac{\beta+1}{\beta+\gamma}}\delta^{\frac{1}{\beta+\gamma}}}$. From Eq.(10), in ER networks the distribution of cascade failure size triggered by one node follows the Poisson distribution, it means the values of C are relatively homogeneous, so we can approximate it furthermore as

$$p^{C-\alpha C^{\frac{\beta+1}{\beta+\gamma}}\delta^{\frac{1}{\beta+\gamma}}} \approx p^{\langle C\rangle - \alpha\langle C^{\frac{\beta+1}{\beta+\gamma}}\rangle\delta^{\frac{1}{\beta+\gamma}}}. \tag{13}$$

The term of $\sum_{y=0}^{\infty}\frac{C^{\ y}p^y}{y!}$ in Eq.(12) is the taylor expansion of $e^{p\ C}$. Hence, with Eq.(13) one gets the cascade failure size $\omega(1-p)$ of ER networks when a fraction $1-p$ of nodes is perturbed as

$$\omega(1-p) \approx 1 - pe^{(p-1)\ C}\, p^{\langle C\rangle - \alpha\langle C^{\frac{\beta+1}{\beta+\gamma}}\rangle\delta^{\frac{1}{\beta+\gamma}}}. \tag{14}$$

Table S1. The parameters

Dynamics	β	γ
$\mathcal{B}$	0	0
$\mathcal{BD}$	0	$^3/_2$
$\mathcal{E}$	1	1
$\mathcal{R}$	1	0

Values of the parameters used in the considered dynamical models.

Now we apply the method to calculate the cascade failure size approximately for single networks into the dynamical models above. For R dynamics, substituting $\beta = 1, \gamma = 0$ into Eq.(14), we get the cascade failure size of ER networks when a fraction $1-p$ of nodes is perturbed for R dynamics as

$$\omega_R(1-p) = 1 - pe^{(p-1)\langle C \rangle} p^{<C> - \alpha\delta <C^2>}. \quad [15]$$

For BD dynamics, substituting $\beta = 0, \gamma = \frac{3}{2}$ into Eq.(14), we get the cascade failure size of ER networks when a fraction $1-p$ of nodes is perturbed for BD dynamics as

$$\omega_{BD}(1-p) = 1 - pe^{(p-1)\langle C \rangle} p^{\langle C \rangle - \alpha\delta^{\frac{2}{3}} <C^{\frac{2}{3}}>}. \quad [16]$$

For E dynamics, substituting $\beta = 1, \gamma = 1$ into Eq.(14), we get the cascade failure size of ER networks when a fraction $1-p$ of nodes is perturbed for E dynamics as

$$\omega_E(1-p) = 1 - pe^{(p-1)\langle C \rangle} p^{\langle C \rangle - \alpha\delta^{\frac{1}{2}} \langle C \rangle}. \quad [17]$$

To calculate the cascade failure size of the R, BD, and E dynamics from Eqs.(15)-(17), we should get the mapping coefficient α between the real cascading failure size and the node degree. It should be noted that for B dynamics the cascade failure size ω_B is not about the mapping coefficient α, we can calculate the failure size by Eq.(9). We here give two ways to calculate the mapping coefficient α.

The first way is an approximate computational method based on the relationship between $\langle C \rangle$ and moments of k from Eq.(7). According to the relationship between degree distribution of ER networks and cascade failure size of general dynamical behaviors, we can get

$$\langle C \rangle = \sum_k \alpha^{-\frac{\beta+\gamma}{\beta+1}} \delta^{-\frac{1}{\beta+1}} k^{\frac{\beta+\gamma}{\beta+1}} \cdot \frac{e^{-\langle k \rangle} \langle k \rangle^k}{k!}, \quad [18]$$

from which, one obtains the average failure size caused by one perturbed node for R, BD, and E dynamics respectively. Hence, we can estimate the mapping coefficient α correspondingly as

$$\alpha = \delta^{-\frac{1}{\beta+\gamma}} < k^{\frac{\beta+\gamma}{\beta+1}} >^{\frac{\beta+1}{\beta+\gamma}} \langle C \rangle^{-\frac{\beta+1}{\beta+\gamma}}. \quad [19]$$

Here, the parameters of the four models are listed in Table S1. For B dynamics, the average failure size caused by one node is $\langle C \rangle \quad \delta^{-1}$.

The second way is a least-square method of the degree sequence $\{k_i\}$ and the cascade size sequence $\{\delta^{\frac{1}{\beta+\gamma}} C_i^{\frac{\beta+1}{\beta+\gamma}}\}$, which can be seen from Eq.(7). It should be noted that we estimate the mapping coefficient α with first order linear regression, and there is no constant term with our least-square method. Hence, the mapping coefficient α can be written as

$$\alpha = \frac{\sum_{i=1}^{n} k_i \delta^{\frac{1}{\beta+\gamma}} C_i^{\frac{\beta+1}{\beta+\gamma}}}{\sum_{i=1}^{n} \delta^{\frac{2}{\beta+\gamma}} C_i^{2\cdot\frac{\beta+1}{\beta+\gamma}}}. \quad [20]$$

The cascade failure size $\omega(1-p)$ *vs* the fraction $1-p$ of perturbed nodes in ER networks for R, BD, E, and B dynamics respectively are presented in Fig.S1. We compare the results of the above two methods with the theoretical solutions (from Eq.(9)) and simulation results, found that the solutions of our method match well with the simulation results. However, we emphasize that the Eq.(9) is based on the assumption that the failure size of each node is independent with each other. The dynamical behaviors are always coupled together, especially when a fraction of $1-p$ of nodes is perturbed simultaneously. We should also see that too large or too small values of tolerance coefficient, δ, may lead to the relationship between C_i and k_i not valid, causing our method out of effect. Notice that the effective range of our method should be explored furthermore in our future works so that the best estimation of the mapping coefficient α is to be picked. Despite this, our method to estimate the cascade failure size caused by a fraction of $1-p$ of nodes in single networks can help us to explore more complex failure situations in interdependent networks.

Universality of spreading dynamics in interdependent networks. When considering ER-ER interdependent-networks, the fraction of nodes in the giant components of A and B at the end of the cascade process are given by

$$P_{\infty,A} = e^{-f_B q_A \;\; C} \left(1 - f_B q_A\right)^{<C> - \alpha <C \;\; > \delta^{\frac{1}{\beta+\gamma}} + 1} \left(1 - f_A\right) p \, , \qquad [21]$$

$$P_{\infty,B} = e^{-q_B \left(1 - p \; (1 - f_A)\right) \; C} \left[1 - q_B \left[1 - p \; (1 - f_A)\right]\right]^{<C> - \alpha <C \;\; > \delta^{\frac{1}{\beta+\gamma}} + 1} \left(1 - f_B\right). \qquad [22]$$

Here, $\;\; = \frac{\beta+1}{\beta+\gamma}$, k_A and k_B represent the average degree of network A and B. f_A and f_B can be solved by

$$e^{-f_B q_A \;\; C} \left(1 - f_B q_A\right)^{<C> - \alpha <C \;\; > \delta^{\frac{1}{\beta+\gamma}} + 1} - \frac{\tau_A}{p} = 0, \qquad [23]$$

$$e^{-q_B \left(1 - p \; (1 - f_A)\right) \; C} \left[1 - q_B \left[1 - p \; (1 - f_A)\right]\right]^{<C> - \alpha <C \;\; > \delta^{\frac{1}{\beta+\gamma}} + 1} - \tau_B = 0. \qquad [24]$$

By comparing the results of Figs. S2, S3 and S4, one concludes that the systems are more vulnerable (first order transitions occur for a smaller fraction of initially perturbed nodes) when we consider the spreading dynamics. The average degree k and the critical value δ are positively correlated to the robustness in Figs. S2(b), S3(b), S4(b), S2(d), S3(d) and S4(d). When decreasing either the connectivity of the networks or the value of δ, first order phase transitions occur more frequently. Figs. S2(c), S3(c), and S4(c) show that when q_B increases, less removed nodes or weaker dependency strengths are needed for the occurrence of first order phase transitions.

For different dynamical models (in ER-ER interdependent-networks), Eqs.(21) and (22) suffer the following adjustments:

$$P_{\infty,A} = e^{-f_B q_A \;\; C} \left(1 - f_B q_A\right)^{<C> - \alpha <C \;\; A > \delta^{\frac{1}{\beta_A+\gamma_A}} + 1} \left(1 - f_A\right) p \, , \qquad [25]$$

$$P_{\infty,B} = e^{-q_B \left(1 - p \; (1 - f_A)\right) \; C} \left[1 - q_B \left[1 - p \; (1 - f_A)\right]\right]^{<C> - \alpha <C \;\; B > \delta^{\frac{1}{\beta_B+\gamma_B}} + 1} \left(1 - f_B\right). \qquad [26]$$

Here, f_A and f_B can be solved by

$$e^{-f_B q_A \;\; C} \left(1 - f_B q_A\right)^{<C> - \alpha <C \;\; A > \delta^{\frac{1}{\beta_A+\gamma_A}} + 1} - \frac{\tau_A}{p} = 0, \qquad [27]$$

$$e^{-q_B \left(1 - p \; (1 - f_A)\right) \; C} \left[1 - q_B \left[1 - p \; (1 - f_A)\right]\right]^{<C> - \alpha <C \;\; B > \delta^{\frac{1}{\beta_B+\gamma_B}} + 1} - \tau_B = 0. \qquad [28]$$

Here, $\;_A$ and β_A are the parameters used in network A, $\;_B$ and β_B are the parameters used in network B (see Table S1).

Finally, in Fig. S5 we report simulative results of interdependent networks displaying different dynamical behaviors and structures. The size of the network is 500 in all our simulations. The average degree of ER networks is 5, whereas the average degree of Barabási-Albert (BA) networks is 4. The results show that first order percolation transitions not only occur in ER-ER configurations but also occur in ER-BA, BA-ER, BA-BA arrangements. At the same time, in all cases, incorporation of the nodes' dynamics always accelerates the change from a second to a first order percolation transition.

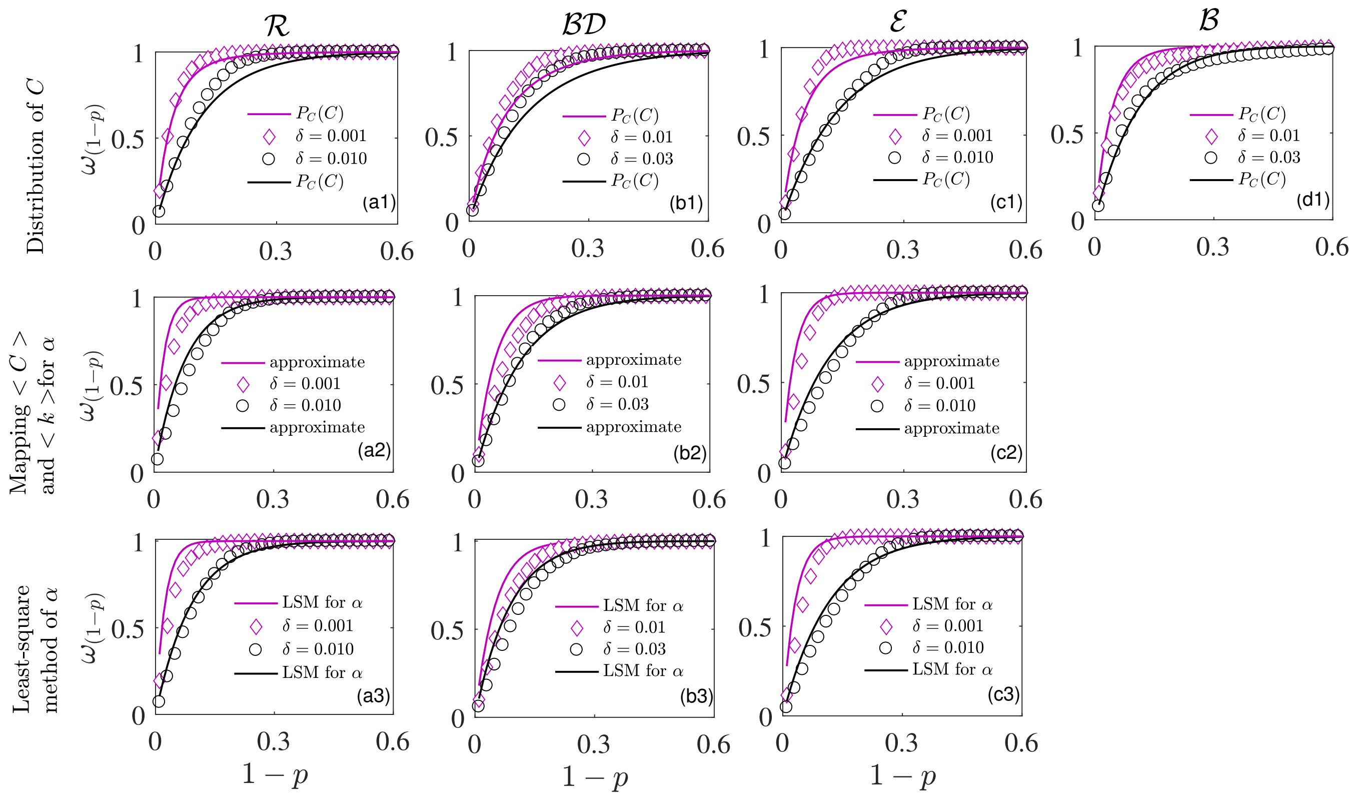

Fig. S1. The cascade failure size $\omega(1-p)$ vs the fraction $1-p$ of perturbed nodes in ER networks for R, BD, E, and B dynamics respectively. (a1),(b1),(c1),and (d1) Comparison of simulation results and theoretical solutions for the four models with different values of tolerance coefficient δ. The theoretical solutions are calculated from Eq.(9) with the distribution $P_C(C)$ of cascade failure size caused by one node. (a2), (b2), and (c2) Comparison of simulation results and approximate theoretical results for R, BD, and E dynamics. The approximate theoretical solutions are calculated from Eqs.(15)-(17), in which the mapping coefficient α is estimated with the relationship between C and moments of k in Eq.(19). It should be noted that we found this approximate method is effective when the tolerance coefficient δ in some certain ranges. For example, for R dynamics with $< k >= 5$ in ER networks the approximate method works well with $0.0005 < \delta < 0.1$; for BD dynamics with $< k >= 2$ in ER networks it works well with $0.002 < \delta < 0.25$; and for E dynamics with $< k >= 5$ in ER networks it works well with $0.0005 < \delta < 0.03$. The reason lies in that the tolerance coefficient δ which is too large or too small will lead to the relationship between C_i and k_i not valid, which enables the approximate method out of effect. (a3), (b3), and (c3) Comparison of simulation results and approximate theoretical results for R, BD, and E dynamics. The approximate theoretical solutions are calculated from Eqs.(15)-(17) as well, but in which the mapping coefficient α is estimated with the least-square method (LSM) of the degree sequence $\{k_i\}$ and the cascade size sequence $\{\delta^{\frac{1}{\beta+\gamma}} C_i^{\frac{\beta+1}{\beta+\gamma}}\}$ from Eq.(20). As for B, there is not the parameter of α, so LSM can not apply to B dynamics.

References

1. Barzel B, Barabási AL (2013) Universality in network dynamics. *Nat. Phys.* 9(10):673–681.

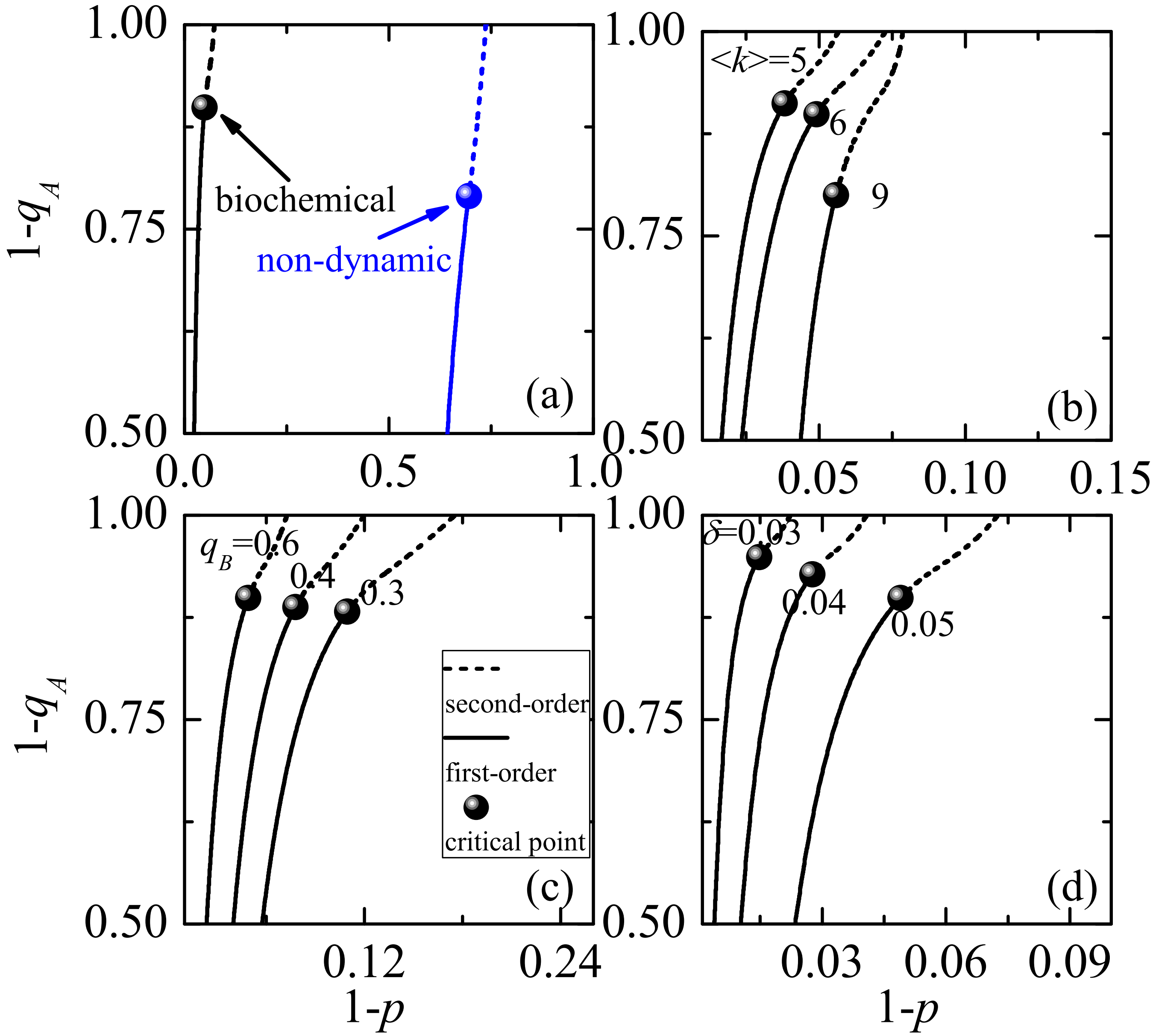

Fig. S2. *B*-*B* model in ER-ER interdependent-networks. (a) Comparison of percolation phase transition of ER-ER networks with *B*-*B* model (black line) and without dynamics (blue line). Solid (dashed) lines indicate first (second) order phase transitions, and ball indicates the critical point. (b), (c), and (d) show the contribution of three factors to the system failure mode including the the average degree (k =5, 6, and 9 from left to right), interdependence strengths (q_B =0.6, 0.4, and 0.3 from left to right), and the tolerance coefficient (δ=0.03, 0.04, and 0.05 from left to right).

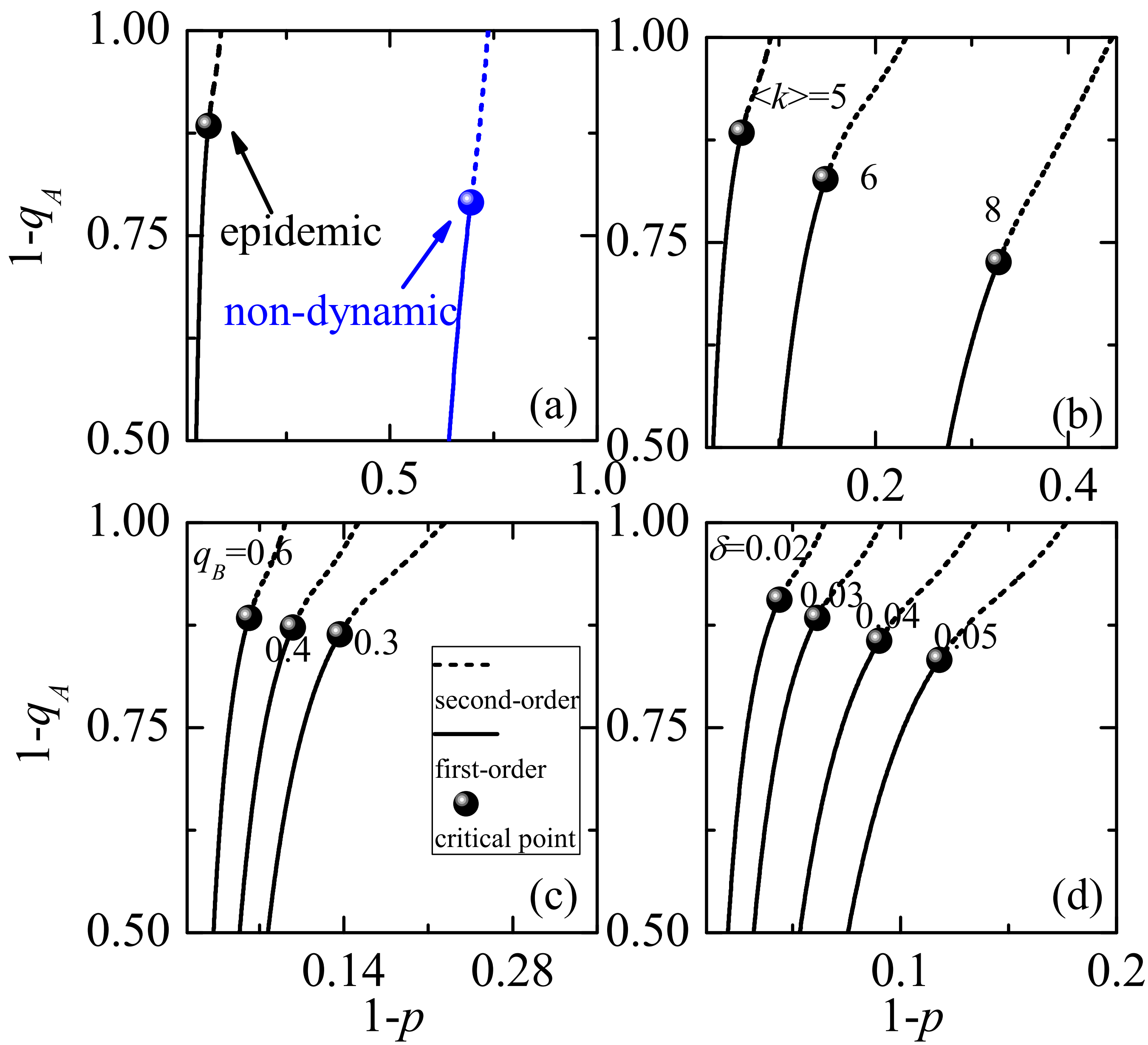

Fig. S3. Percolation phase transition for E-E in ER-ER interdependent-networks. (b), (c), and (d) show the contribution to the system failure mode of tree factors, including the the average degree (k =5, 6, and 8 from left to right), interdependence strengths (q_B =0.6, 0.4, and 0.3 from left to right), and the tolerance coefficient (δ=0.02, 0.03, 0.04, and 0.05 from left to right).

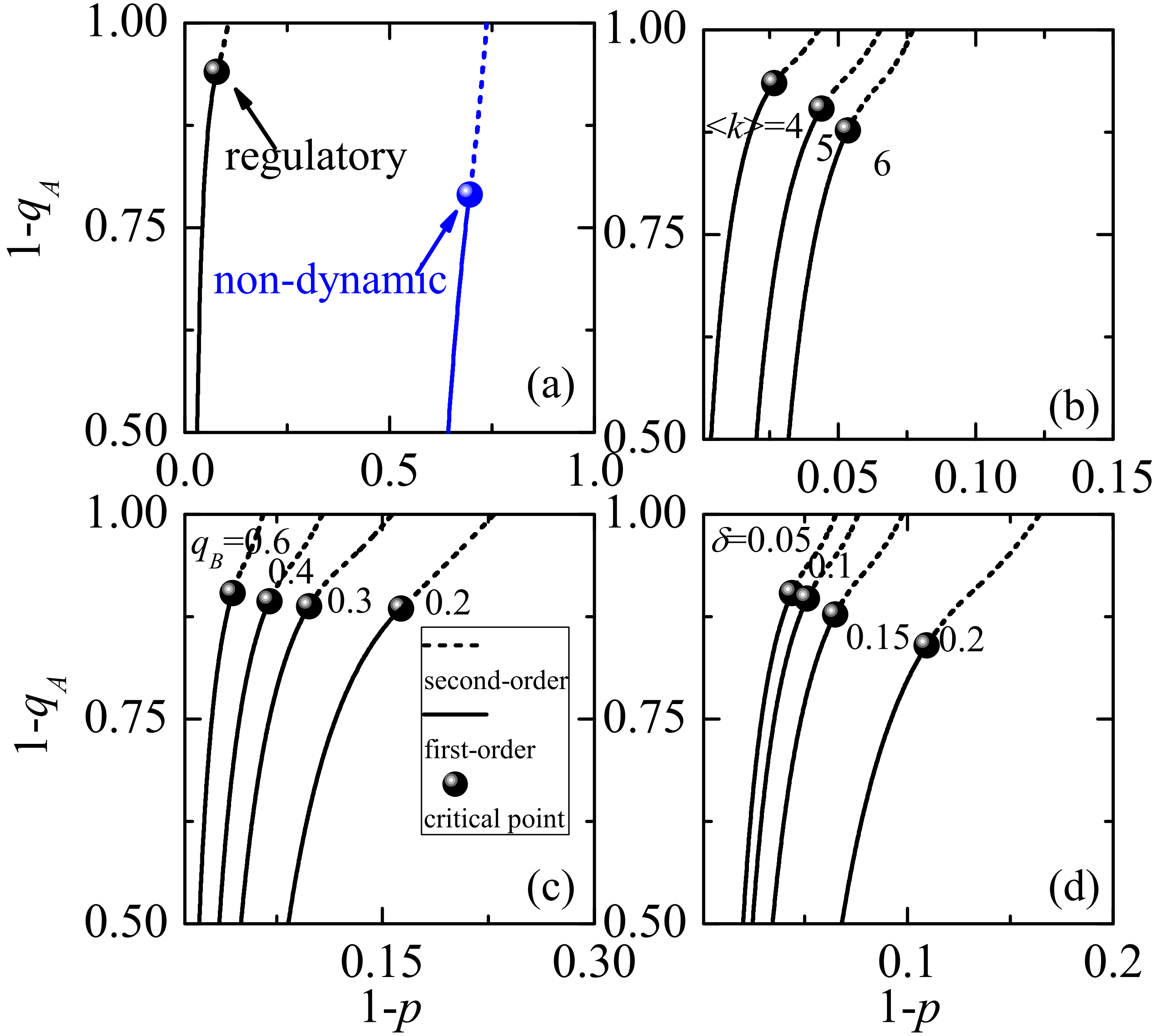

Fig. S4. Percolation transition for R-R in ER-ER interdependent-networks. (b), (c), and (d) show the contribution to the system failure mode of tree factors, including the the average degree (k =4, 5, and 6 from left to right), interdependence strengths (q_B =0.6, 0.4, 0.3, and 0.2 from left to right), and the tolerance coefficient (δ=0.05, 0.1, 0.15, and 0.2 from left to right).