Percolation of partially interdependent networks under targeted attack

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We study a system composed of two partially interdependent networks; when nodes in one network fail, they cause dependent nodes in the other network to also fail. In this paper, the percolation of partially interdependent networks under targeted attack is analyzed. We apply a general technique that maps a targeted-attack problem in interdependent networks to a random-attack problem in a transformed pair of interdependent networks. We illustrate our analytical solutions for two examples: (i) the probability for each node to fail is proportional to its degree, and (ii) each node has the same probability to fail in the initial time. We find the following: (i) For any targeted-attack problem, for the case of weak coupling, the system shows a second order phase transition, and for the strong coupling, the system shows a first order phase transition. (ii) For any coupling strength, when the high degree nodes have higher probability to fail, the system becomes more vulnerable. (iii) There exists a critical coupling strength, and when the coupling strength is greater than the critical coupling strength, the system shows a first order transition; otherwise, the system shows a second order transition.

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I. INTRODUCTION

Complex networks exist in many different areas in the real world and have been studied in the past 15 years. However, almost all researchers have been focused on properties of a single network that does not interact with or depend on other networks [1–11]. Such situations rarely, if ever, occur in reality [12–16]. In 2010, Buldyrev *et al.* [12] developed a theoretical framework for studying the process of cascading failures in fully interdependent networks caused by random initial failure of nodes. Surprisingly, they found a first order percolation transition and that a broader degree distribution increased the vulnerability of interdependent networks to random failure, in contrast to the behavior of a single network. Recently, five important generalizations of the basic model [13–20] have been proposed sequentially. (i) Parshani *et al.* [13] presented a theoretical framework for studying the case of partially interdependent networks. Their findings showed that reducing the coupling strength lead to a change from a first to second order percolation transition. (ii) Because in the real world a network is not always attacked randomly, Huang *et al.* [14] investigated the robustness of fully interdependent networks under targeted attack. The result implied that interdependent networks are difficult to defend. (iii) In real scenarios, the assumption that one node in a network depends only on one node in the other network is not valid. Shao *et al.* [17] investigated a framework to study the percolation of two interdependent networks with multiple support-dependent relations. (iv) Hu *et al.* [18] studied the percolation of a pair of coupled networks with both interdependency links and connectivity links. They found unusual discontinuous changes from second order to first order transition as a function of the dependency coupling between the two networks. (v) In the real word, for more than two networks coupled together, Gao *et al.* [15,19,20] proposed a framework to study the robustness of a network of networks (NON). Their results showed that for a treelike Erdös-Rényi (ER) NON the robustness decreases with the number of networks and for a looplike ER NON the giant component is independent of the number of networks. However, for real scenarios, two infrastructures are always partially coupled together [21,22], such as energy and communications, power stations and transportation, etc., and they might be attacked intentionally on high degree nodes. Understanding the robustness due to partial interdependency and under targeted attack is one of the major challenges for designing resilient infrastructures.

Here we develop a generalized framework to study the percolation of partially interdependent networks under targeted attack. We further develop a general technique [14] that maps the targeted-attack problem in interdependent networks to the random-attack problem in a transformed pair of interdependent networks. We find the following: (i) For any targeted-attack problem, for the case of weak coupling, the system shows a second order phase transition, and for strong coupling, the system shows a first order phase transition. (ii) For any coupling strength, when the high degree nodes have a larger probability to fail, the system becomes more vulnerable. (iii) There exists a critical coupling strength, and when the coupling strength is greater than the critical coupling strength, the system shows a first order transition; otherwise, the system shows a second order transition. In the following two examples, the critical coupling strength can be explicitly derived analytically: (i) the probability for each node to fail is proportional to its degree, and (ii) each node has the same probability to fail in the initial time. Although case (ii) was solved in Ref. [15], we present here a more general case where both interdependent networks are initially attacked randomly.

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II. THE MODEL

In this model, there are two networks, A and B, with the number of nodes N_A and N_B, and within each network, the nodes are connected with degree distributions $P_A(k)$ and $P_B(k)$, respectively. We suppose that the average degree of network A is a and the average degree of network B is b. In addition, a fraction q_A of A nodes depends on the nodes in network B, and a fraction q_B of B nodes depends on the nodes in network A. That is, if node A_i of network A depends on node B_j of network B and B_j depends on node A_s of network A, then $s = i$, which satisfies the no-feedback condition [19]. Consequently, when nodes in one network fail, the interdependent nodes in the other network also fail, and we suppose that only the nodes in the giant component remain functional, which leads to further failure in the first network. This dynamic process leads to a cascade of failures. In order to study the cascading failure under targeted attack, we apply the general technique that a targeted-attack problem in networks can be mapped to a random-attack problem [14,23]. A value $W_\alpha(k_i)$ is assigned to each node, which presents the probability that a node i with k_i links becomes inactive by targeted attack. We focus on the family of functions [24]

$$W_\alpha(k_i) = \frac{k_i^\alpha}{\sum_{i=1}^N k_i^\alpha}, \quad -\infty < \alpha < +\infty. \tag{1}$$

When $\alpha > 0$, nodes with a higher degree are more vulnerable to the targeted attack, while for $\alpha < 0$, nodes with a lower degree have a higher probability to fail. For $\alpha = 0$, all the nodes in a network have the same probability to fail, which is equivalent to the case of random attack.

Without loss of generality, we begin by studying the generating function and the giant component of network A after a targeted attack, which can be directly applied to network B. Next we study the generating functions of network A at each iteration step.

(i) The generating function of network A is defined as

$$G_{A0}(x) = \sum_k P_A(k)x^k. \tag{2}$$

The generating function of the associated branching process is $G_{A1}(x) = G'_{A0}(x)/G'_{A0}(1)$ [12,13,25,26]. The average degree of network A is defined as $a = \bar{k} = \sum_k P_A(k)k$.

(ii) We intentionally remove $1 - p_1$ fraction of nodes from network A according to Eq. (1) and remove the links between the removed nodes. Thus, we obtain that the generating function of the nodes left in network A is [14,26,27]

$$G_{Ab}(x) = \sum_k P_A^{p_1}(k)x^k = \frac{1}{p_1}\sum_k P_A(k)h_1^{k^\alpha}x^k, \tag{3}$$

where the new degree distribution of the remaining p_1 fraction of nodes $P_A^{p_1}(k) \equiv \frac{1}{p_1}P_A(k)h_1^{k^\alpha}$, and h_1 satisfies

$$p_1 = G_\alpha(h_1) \equiv \sum_k P_A(k)h_1^{k^\alpha}, \quad h_1 \equiv G_\alpha^{-1}(p_1). \tag{4}$$

The average degree of the remaining nodes in network A in this step is $\bar{k}(p_1) = \sum_k P_A^{p_1}(k)k$.

(iii) We remove the links between the removal nodes and the remaining nodes. Thus we obtain that the generating function of the network composed by the remaining nodes is [27]

$$G_{Ac}(x) = G_{Ab}(1 - \tilde{p}_1 + \tilde{p}_1 x), \tag{5}$$

where $\tilde{p}_1$ is the fraction of the original links that connect to the nodes that remain, which satisfies

$$\tilde{p}_1 = \frac{p_1 N_A \bar{k}(p_1)}{N_A \bar{k}} = \frac{\sum_k P_A(k)kh_1^{k^\alpha}}{\sum_k P_A(k)k}. \tag{6}$$

Then we can find the equivalent network A' with generating function $\tilde{G}_{A0}(x)$, such that after a fraction $1 - p_1$ of nodes is randomly removed, the new generating function of nodes left in A' is the same as $G_{Ac}(x)$. By solving the equation $\tilde{G}_{A0}(1 - p_1 + p_1 x) = G_{Ac}(x)$ and Eq. (5), we can get

$$\tilde{G}_{A0}(x) = G_{Ab}\left(1 - \frac{\tilde{p}_1}{p_1} + \frac{\tilde{p}_1}{p_1}x\right). \tag{7}$$

And the generating function of the associated branching process $\tilde{G}_{A1}(x) = \tilde{G}'_{A0}(x)/\tilde{G}'_{A0}(1)$.

(iv) Thus, the targeted-attack problem on network A can be mapped to the random-attack problem on network A'. For network A, a $1 - p_1$ fraction of nodes in network A is intentionally removed according to Eq. (1), and the fraction of nodes that belongs to the giant component is [14,27,28]

$$p_A(p_1) = 1 - \tilde{G}_{A0}[1 - p_1(1 - f_A)], \tag{8}$$

where $f_A \equiv f_A(p_1)$ satisfies a transcendental equation,

$$f_A = \tilde{G}_{A1}[1 - p_1(1 - f_A)]. \tag{9}$$

For network B, a $1 - p_2$ fraction of nodes in network B is intentionally removed according to Eq. (1), and the fraction of nodes that belongs to the giant component $p_B(p_2)$ is similar to Eq. (8), but p_1 changes to p_2 and A changes to B.

According to the definition of the fraction of nodes that belongs to the giant component, we perform the dynamic of cascading failures as follows: Initially, the $1 - p_1$ and $1 - p_2$ fractions of nodes are intentionally removed from network A and network B, respectively. The remaining fraction of network A nodes after an initial removal of $1 - p_1$ is $\psi'_1 = p_1$, and the remaining fraction of network B nodes after an initial removal of $1 - p_2$ is $\phi'_0 = p_2$. The remaining functional part of network A contains a fraction $\psi_1 = \psi'_1 p_A(\psi'_1)$ of network nodes. Accordingly, for the same reason, the remaining fraction of network B is $\phi'_1 = p_2\{1 - q_B[1 - p_A(\psi'_1)p_1]\}$, and the fraction of nodes in the giant component of network B is $\phi_1 = \phi'_1 p_B(\phi'_1)$. Then the sequence, ψ_n and ϕ_n, of giant components and the sequence, ψ'_n and ϕ'_n, of the remaining fractions of nodes at each stage of the cascading failures are constructed as follows:

$$\begin{aligned}
&\psi'_1 = p_1, \quad \psi_1 = \psi'_1 p_A(\psi'_1), \quad \phi'_0 = p_2,\\
&\phi'_1 = p_2\{1 - q_B[1 - p_A(\psi'_1)p_1]\}, \quad \phi_1 = \phi'_1 p_B(\phi'_1),\\
&\psi'_2 = p_1\{1 - q_A[1 - p_B(\phi'_1)p_2]\}, \quad \psi_2 = \psi'_2 p_A(\psi'_2),\\
&\phi'_2 = p_2\{1 - q_B[1 - p_A(\psi'_2)p_1]\}, \quad \phi_2 = \phi'_2 p_B(\phi'_2), \\
&\quad\ldots\\
&\psi'_n = p_1\{1 - q_A[1 - p_B(\phi'_{n-1})p_2]\}, \quad \psi_n = \psi'_n p_A(\psi'_n),\\
&\phi'_n = p_2\{1 - q_B[1 - p_A(\psi'_n)p_1]\}, \quad \phi_n = \phi'_n p_B(\phi'_n).
\end{aligned} \tag{10}$$

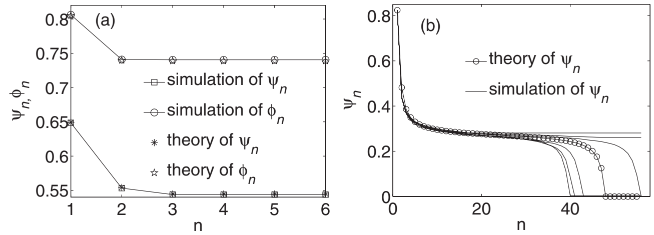

FIG. 1. (a) The giant component of both networks A and B, ψ_n and ϕ_n, after time n cascading failures for the case when $a = b = 3$, $p_1 = 0.8$, $p_2 = 0.9 > p_2^c$, $q_A = 0.45$, $\alpha = 1$, and $q_B = 0.15$. The simulation results show excellent agreement with the theory, system (10). All estimates are the results of averaging over 40 realizations. (b) The giant component of network A, ψ_n, after time n cascading failures for the case when $a = b = 3$, $p_1 = 0.9$, $q_A = 0.65$, $q_B = 0.7$, $\alpha = 0$, and $p_2 = 0.6726 < p_2^c = 0.673$. The simulation results show excellent agreement with the theory, system (10). In both (a) and (b), $N_A = N_B = 2 \times 10^5$.

Figure 1 shows the giant components ψ_n and ϕ_n as functions of time step n for different values of $a = b$, p_1, p_2, q_A, q_B, and α. The simulation results show excellent agreement with the theory, system (10). Figure 1(a) shows that a finite giant component exists for $p_2 > p_2^c$, and Fig. 1(b) shows for the case when $p_2 < p_2^c$, the two networks collapse.

Next, we study the steady state of system (10) after the cascading failures, which can be represented by ψ'_n, ϕ'_n at the limit of $n \to \infty$. The limit must satisfy the equations $\psi'_n = \psi'_{n+1}, \phi'_n = \phi'_{n+1}$ since eventually the clusters stop fragmenting and the fractions of randomly removed nodes at steps n and $n+1$ are equal. Denoting $\psi'_n = x$, $\phi'_n = y$, we arrive at a system of two symmetric equations:

$$x = p_1\{1 - q_A[1 - p_B(y)p_2]\},$$
$$y = p_2\{1 - q_B[1 - p_A(x)p_1]\}. \tag{11}$$

III. ANALYTICAL SOLUTION

In this section we present two examples that can be explicitly solved analytically: (i) $\alpha = 1$ and (ii) $\alpha = 0$ for two interdependent ER networks. Case (ii) is similar to that of Parshani *et al.* [13] but more general. For the ER [29,30] networks, characterized by the Poisson degree distribution, $G_{A0}(x) = G_{A1}(x) = \exp[a(x-1)]$, $G_{B0}(x) = G_{B1}(x) = \exp[b(x-1)]$.

(i) For the case of $\alpha = 1$, substituting $\alpha = 1$ into Eqs. (3)–(7), we obtain that $G_{Ab}(x)$, $G_{Ac}(x)$, and $\tilde{G}_{A0}(x)$ can be represented by $G_{A0}(x)$, which reflects the mapping from a targeted-attack problem to random-attack problem. Then we get $\tilde{G}_{A0}(x) = \tilde{G}_{A1}(x) = \exp[ah_1^2(x-1)]$, $\tilde{G}_{B0}(y) = \tilde{G}_{B1}(y) = \exp[bh_2^2(y-1)]$. Thus, from Eq. (9) we obtain

$$f_A = \exp\left[-ah_1^2 x(1-f_A)\right], \quad f_B = \exp\left[-bh_2^2 y(1-f_B)\right]. \tag{12}$$

Substituting Eqs. (8), (9), and (11) into Eqs. (12), by eliminating x and y, we obtain

$$f_A = e^{-ap_1h_1^2(1-f_A)[1-q_A+p_2q_A(1-f_B)]},$$
$$f_B = e^{-bp_2h_2^2(1-f_B)[1-q_B+p_1q_B(1-f_A)]}. \tag{13}$$

According to the definition of $\psi_\infty = p_A(x)x$ and $\phi_\infty = p_B(y)y$, we obtain the giant component of networks A and B, respectively, at the end of the cascading failure as

$$\psi_\infty = p_1(1-f_A)[1-q_A+p_2q_A(1-f_B)],$$
$$\phi_\infty = p_2(1-f_B)[1-q_B+p_1q_B(1-f_A)]. \tag{14}$$

Solving the Eqs. (13), we obtain f_A and f_B, and then we obtain ψ_∞ and ϕ_∞ by substituting f_A and f_B into Eqs. (14).

The numerical simulation results of system (14) are shown in Fig. 2. As shown in Fig. 2, for fixed a, b, and q_B, there exists a critical p_2^c; when $p_2 < p_2^c, \phi_\infty = 0$, and when $p_2 > p_2^c$, $\phi_\infty > 0$. For the weak coupling case, i.e., when q_A is small ($q_A = 0.1$ in Fig. 2), $\phi_\infty(p_2^c) = 0$, which shows a second order phase transition, and the transition threshold is defined as p^{II}. For strong coupling, i.e., when q_A is large ($q_A = 0.7$ in Fig. 2), $\phi_\infty(p_2^c) > 0$, which represents a first order percolation

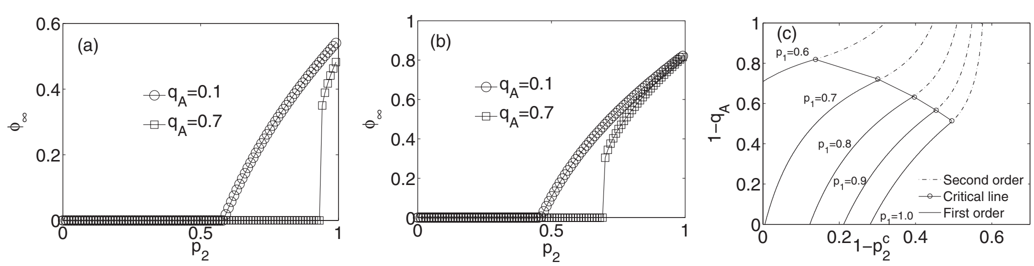

FIG. 2. (a) The giant component ϕ_∞ of network B as a function of the initial attack on network B, $1 - p_2$, when $p_1 = 0.7$, $a = 3$, $b = 4$, $q_B = 0.7$, and $\alpha = 1$ for two different q_A. (b) The giant component ϕ_∞ of network B as a function of the initial attack on network B, $1 - p_2$, when $p_1 = 0.9$, $a = 3$, $b = 4$, $q_B = 0.7$, and $\alpha = 1$ for two different q_A. For the weak coupling strength ($q_A = 0.1$), the system shows a second order phase transition, and for the strong coupling strength ($q_A = 0.7$), the system shows a first order phase transition. From (a) and (b), we find that the changes in the critical threshold depend on the changes in p_1. (c) The coupling strength $1 - q_A$ as a function of $1 - p_2^c$ for different values of the remaining fraction of nodes after the initial attack on network A, p_1, when $a = 3$, $b = 4$, $q_B = 0.7$. For each p_1, $1 - q_A$ as a function of $1 - p_2^c$ is divided into two regions by an open circle. The dash-dotted curve above an open circle represents the second order phase transition, and the solid curve below the open circle represents the first order phase transition. All the circles are connected to form a critical line, which represents $1 - q_A^c$ as a function of $1 - p_2^c$. It also shows that q_A^c increases as p_1 increases.

phase transition, and the transition threshold is defined as p^I. Figures 2(a) and 2(b) indicate that there exists a critical q_A^c, which corresponds to the condition when $p^I = p^{II}$; when $q_A < q_A^c$, the system shows a second order phase transition, and when $q_A > q_A^c$, the system shows a first order phase transition. Furthermore, Figs. 2(a) and 2(b) indicate that the critical threshold changes with p_1, i.e., q_A^c also changes with p_1. In Fig. 2(c), we studied by numerical simulation $1 - q_A$ as a function of $1 - p_2^c$ for different values of p_1 when $a = 3$, $b = 4$, $q_B = 0.7$. As shown in Fig. 2(c), for each p_1, there exists a critical q_A^c (open circles), which corresponds to the condition $p^I = p^{II}$. Moreover, q_A^c increases as p_1 increases, which is represented by the curve connecting the circles in Fig. 2(c) and indicates that the two networks become more robust as q_A decreases.

Next, we study the transition threshold p^I and p^{II} analytically when $a = b = \bar{k}$, $p_1 = p_2 = p$, $q_A = q_B = q$. In this case, from Eqs. (13) and (14), we obtain that the giant components of networks A and B at the end of the cascading failure $\psi_\infty = \phi_\infty$ satisfy

$$\phi_\infty = p\left(1 - e^{-\bar{k}h^2\phi_\infty}\right)\left[1 - q + pq\left(1 - e^{-\bar{k}h^2\phi_\infty}\right)\right], \quad (15)$$

and $f \equiv f_A = f_B$ satisfies

$$f = e^{-\bar{k}ph^2(1-f)[1-q+pq(1-f)]}, \quad (16)$$

where $h = \ln p/\bar{k} + 1$. The condition for the first order transition ($p = p^I$) is

$$1 = f[\bar{k}p^I h^2(1-q) + 2\bar{k}(p^I)^2 qh^2(1-f)], \quad 0 \leqslant f < 1. \quad (17)$$

And solving Eq. (16) for $f \to 1$ yields the condition for the second order transition ($p = p^{II}$),

$$\bar{k}p^{II}(1-q)h^2 = 1. \quad (18)$$

The analysis of Eqs. (17) and (18) shows the first order transition at $p = p^I$ occurs for networks with strong coupling ($q > q_c$), whereas the second order transition at $p = p^{II}$ occurs for networks with weak coupling ($q < q_c$). This behavior is shown in Fig. 3, where the solid curves show the case of first order phase transition and the dash-dotted curves show the case of second order phase transition. The critical value of q_c (and p_c) for which the phase transition changes from first order to second order is obtained when the conditions for both the first and second order transitions are satisfied simultaneously. Applying both conditions, Eqs. (17) and (18), we obtain

$$\bar{k}\left[1 + \ln\left(\frac{1-q_c}{2q_c}\right)\Big/\bar{k}\right]^2 = \frac{2q_c}{(1-q_c)^2}. \quad (19)$$

Solving Eq. (19), we obtain q_c, and then we can get p_c by

$$p_c = \frac{1-q_c}{2q_c}. \quad (20)$$

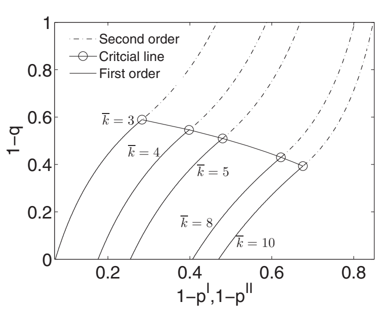

FIG. 3. The coupling strength $1 - q = 1 - q_A = 1 - q_B$ as a function of the first order and second order phase transition thresholds, $1 - p^I, p^{II}$ for different values of average degree $\bar{k} = a = b$ when $\alpha = 1$. The curve connecting the circles shows the critical line, below which the system shows a first order phase transition and above which the system shows a second order phase transition. The simulation of the critical line agrees well with the theory, Eq. (20).

(ii) For the case when $\alpha = 0$, $W_0 = 1/N$, the targeted-attack problem is equivalent to the random-attack problem studied in Ref. [13]. For the case of two ER [29,30] networks with average degrees a and b, we can easily get $p_A(x) = 1 - f_A$, $p_B(y) = 1 - f_B$ from the Eqs. (8) and (9), and system (11) becomes

$$\begin{aligned} x &= p_1[1 - q_A + p_2 q_A(1 - f_B)],\\ y &= p_2[1 - q_B + p_1 q_B(1 - f_A)]. \end{aligned} \quad (21)$$

According to Eqs. (9) and (21), f_A, f_B satisfy

$$\begin{aligned} f_A &= e^{-ap_1(1-f_A)[1-q_A+p_2q_A(1-f_B)]},\\ f_B &= e^{-bp_2(1-f_B)[1-q_B+p_1q_B(1-f_A)]}. \end{aligned} \quad (22)$$

Thus, we obtain the fraction of nodes in the giant components of networks A and B at the end of the cascading process,

$$\begin{aligned} \psi_\infty &= p_1(1-f_A)[1 - q_A + p_2 q_A(1 - f_B)],\\ \phi_\infty &= p_2(1-f_B)[1 - q_B + p_1 q_B(1 - f_A)]. \end{aligned} \quad (23)$$

Our framework is equivalent to Ref. [13] when $p_2 = 1$. In detail, when $p_2 = 1$, Eqs. (22) are the same as Eqs. (7) in Ref. [13]. Here we study the more general case where $p_2 < 1$.

For the case $\alpha = 0$, numerical simulation results of system (23) are shown in Fig. 4. For given $a = b$, q_B, p_1, there exists a critical p_2^c; when $p_2 < p_2^c$, $\phi_\infty = 0$, and when $p_2 > p_2^c$, $\phi_\infty > 0$. For weak coupling, i.e., when q_A is small ($q_A = 0.1$ in Fig. 4), $\phi_\infty(p_2^c) = 0$, representing a second order phase transition, and the percolation threshold is defined as p^{II}. For strong coupling, i.e., when q_A is large (e.g., $q_A = 0.65$ in Fig. 4), $\phi_\infty(p_2^c) > 0$, representing a first order phase transition, and the percolation threshold is defined as p^I. Similar to the case when $\alpha = 1$, Fig. 4 indicates that there exists a critical q_A^c, which corresponds to the condition when $p^I = p^{II}$. When $q_A < q_A^c$, the system shows a second order phase transition, and when $q_A > q_A^c$, the system shows a first order phase transition. Furthermore, Fig. 4 indicates that the critical threshold changes with p_1, i.e., q_A^c also changes with p_1. In Fig. 4(c), we investigate, using numerical calculations, $1 - q_A$ as a function of $1 - p_2^c$ for different values of p_1 when $a = b = 3$, $q_B = 0.7$. As shown in Fig. 4(c), for each p_1, there exists a critical q_A^c (open circles), which corresponds to the condition $p^I = p^{II}$. Moreover, q_A^c increases as p_1 increases, which is represented by the curve with circles in Fig. 4(c) and

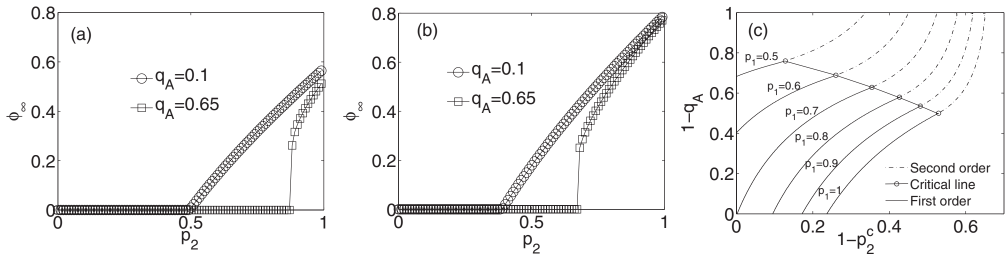

FIG. 4. (a) The giant component ϕ_∞ of network B as a function of the initial attack on network B, $1-p_2$, when $p_1=0.7$, $a=b=3$, $q_B=0.7$, and $\alpha=0$ for two different q_A. (b) The giant component ϕ_∞ of network B as a function of the initial attack on network B, $1-p_2$, when $p_1=0.9$, $a=b=3$, $q_B=0.7$, and $\alpha=0$ for two different q_A. For the weak coupling strength ($q_A=0.1$), the system shows a second order phase transition, and for the strong coupling strength ($q_A=0.65$), the system shows a first order phase transition. (c) The coupling strength $1-q_A$ as a function of $1-p_2^c$ for different values of the remaining fraction of nodes after the initial attack on network A, p_1, when $a=b=3$, $q_B=0.7$. For each p_1, $1-q_A$ as a function of $1-p_2^c$ is divided into two regions by open circles. The dash-dotted curve above an open circle represents the second order phase transition, and the solid curve below an open circle represents the first order phase transition. All the circles are connected to form a critical line, which represents $1-q_A^c$ as a function of $1-p_2^c$. It also shows that q_A^c increases as p_1 increases.

indicates that the two networks become more robust as q_A decreases.

By substituting $a=b=\bar{k}$, $p_1=p_2=p$, $q_A=q_B=q$ into Eqs. (22) and (23), we obtain that the giant components of networks A and B at the end of the cascading failure $\psi_\infty=\phi_\infty$ satisfy

$$\phi_\infty = p(1-e^{-\bar{k}\phi_\infty})[1-q+pq(1-e^{-\bar{k}\phi_\infty})], \qquad (24)$$

and $f\equiv f_A=f_B$ satisfies

$$f = e^{\bar{k}p(f-1)[1-q+pq(1-f)]}, \quad 0 \leqslant f < 1. \qquad (25)$$

Thus we obtain the condition for the first order transition ($p=p^I$),

$$1 = f[\bar{k}p^I(1-q) + 2\bar{k}(p^I)^2 q(1-f)]. \qquad (26)$$

Solving Eq. (25) for $f\to 1$ yields the condition for the second order transition ($p=p^{II}$),

$$\bar{k}p^{II}(1-q) = 1. \qquad (27)$$

Similar to the case of $\alpha=1$, the analysis of Eqs. (26) and (27) shows that the first order transition at $p=p^I$ occurs for networks with strong coupling ($q>q_c$), whereas the second order transition at $p=p^{II}$ occurs for networks with weak coupling ($q<q_c$). This behavior is shown in Fig. 5, where the solid curves show the case of first order phase transition, and the dashed-dotted curves show the case of second order phase transition. The critical values of q_c (and p_c) for which the phase transition changes from first order to second order are obtained when the conditions for both the first and second order transitions are satisfied simultaneously. Applying both conditions Eqs. (26) and (27), we obtain

$$q_c = \frac{\bar{k}+1-\sqrt{2\bar{k}+1}}{\bar{k}}, \quad p_c = \frac{\sqrt{2\bar{k}+1}+1}{2\bar{k}}. \qquad (28)$$

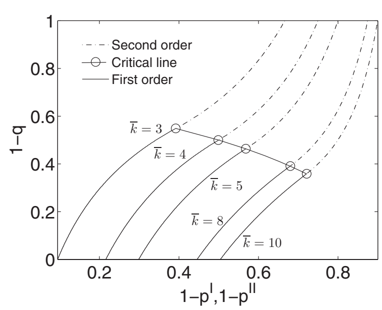

FIG. 5. The coupling strength $1-q=1-q_A=1-q_B$ as a function of the first order and second order phase transition thresholds, $1-p^I$, $1-p^{II}$, for different values of average degree $\bar{k}=a=b$ when $\alpha=0$. The curve with circles shows the critical line, below which the system shows a first order phase transition, and above which the system shows a second order phase transition. The simulation of the critical line agrees well with the theory, Eq. (28).

IV. NUMERICAL SOLUTIONS OF THE GENERAL CASE

Our theoretical study can be applied to any case of α. In this section, we investigate the solutions for the general cases of α. Figure 6 shows the giant component ϕ_∞ of network B as a function of the initial attack on network B, $1-p_2$

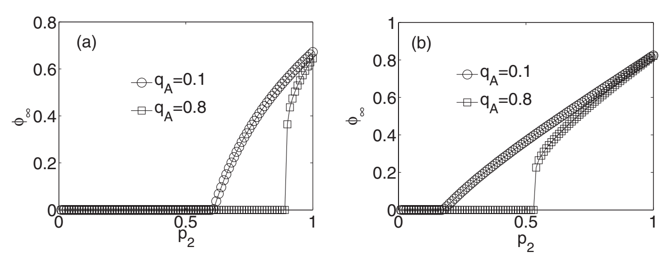

FIG. 6. (a) The giant component ϕ_∞ of network B as a function of the initial attack on network B, $1-p_2$, when $p_1=0.8$, $a=3$, $b=4$, $q_B=0.7$, and $\alpha=2$ for two different q_A. (b) The giant component ϕ_∞ of network B as a function of the initial attack on network B, $1-p_2$, when $p_1=0.8$, $a=3$, $b=4$, $q_B=0.7$, and $\alpha=-1$ for two different q_A. For weak coupling strength ($q_A=0.1$), the system shows a second order phase transition, and for strong coupling strength ($q_A=0.8$), the system shows a first order phase transition.

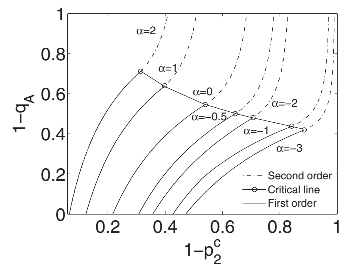

FIG. 7. The coupling strength $1-q_A$ as a function of $1-p_2^c$ for different values of α when $a=3$, $b=4$, $q_B=0.7$, and $p_1=0.8$. The curve connecting the circles shows the critical line, below which the system shows a first order phase transition and above which the system shows a second order phase transition.

for $\alpha=2$ [Fig. 6(a)] and $\alpha=-1$ [Fig. 6(b)]. For given a, b, q_B, p_1, there exists a critical p_2^c; when $p_2<p_2^c$, $\phi_\infty=0$, and when $p_2>p_2^c$, $\phi_\infty>0$. For weak coupling, i.e., when q_A is small ($q_A=0.1$ in Fig. 6), $\phi_\infty(p_2^c)=0$, which shows a second order phase transition. For strong coupling, i.e., when q_A is large ($q_A=0.8$ in Fig. 6), $\phi_\infty(p_2^c)>0$, which shows a first order phase transition. Figure 6 indicates that there exists a critical q_A^c; when $q_A<q_A^c$, it shows a second order phase transition, and when $q_A>q_A^c$, the system shows a first order phase transition. Furthermore, Fig. 6 indicates that the critical threshold changes with α, i.e., q_A^c also changes with α.

In Fig. 7, we investigate the numerical simulation of $1-q_A$ as a function of $1-p_2^c$ for different values of α when $a=3$, $b=4$, $p_1=0.8$, $q_B=0.7$. As shown in Fig. 7, for each α, there exists a critical q_A^c (circles). Moreover, $1-q_A^c$ increases as α increases, which is represented by the curve connecting the circles in Fig. 7 and indicates that the two networks becomes more robust as α decreases.

V. CONCLUSIONS

In summary, we developed a framework for studying the percolation of two partially interdependent ER networks under targeted attack for the cases of a high degrees of attack, $\alpha=1$, and a random attack, $\alpha=0$. For any value of α, the system shows a second order phase transition when q is small, and a first order phase transition when q is large. We find the critical q_c and critical threshold p_c when the percolation of the system changes from first to second order for the cases when $\alpha=1$ and $\alpha=0$. Moreover, we find that when α increases, i.e., the high degree nodes have a larger probability to fail, the system becomes more vulnerable.

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