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Cross-sectional fluctuation scaling in the high-frequency illiquidity of Chinese stocks

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Abstract – Taylor's law of temporal and ensemble fluctuation scaling has been ubiquitously observed in diverse complex systems including financial markets. Stock illiquidity is an important nonadditive financial quantity, which is found to comply with Taylor's temporal fluctuation scaling law. In this paper, we perform the cross-sectional analysis of the 1 min high-frequency illiquidity time series of Chinese stocks and unveil the presence of Taylor's law of ensemble fluctuation scaling. The estimated daily Taylor scaling exponent fluctuates around 1.442. We find that Taylor's scaling exponents of stock illiquidity do not relate to the ensemble mean and ensemble variety of returns. Our analysis uncovers a new scaling law of financial markets and might stimulate further investigations for a better understanding of financial markets' dynamics.

Introduction. – A complex system is composed of many interacting constituents $\{i|i = 1, 2, \cdots, N\}$ that form a complex network. For a given quantity f of the nodes, we can record N time series sampled at evenly spaced time intervals, $\{f_i(t)|t = 1, 2, \cdots, T\}$. A well-studied example in econophysics is about financial markets, in which the listed companies are constituents or nodes and the recoded time series are returns and trading volumes. In addition to the widely adopted analysis based on the random matrix theory [1–3], Taylor's law of fluctuation scaling has also attracted much attention [4]. As pointed out by Eisler *et al.* [4], the fluctuation scaling law was first discovered in 1938 by Smith [5], who found that the variance of the yields of crop fields scales as a power law of the average yield or equivalently the plot size. In 1961, Taylor obtained a similar power-law relationship between the variance and the mean of populations [6], which is later known as Taylor's law of spatial fluctuation scaling.

When time series are considered, there are two forms of Taylor's law, describing the temporal and ensemble fluctuation scaling of a complex system [4]. Taylor's law of temporal fluctuation scaling considers the variance-mean relationship of time series, in which the mean is

$$\mu_i = \langle f_i(t)\rangle_t = \frac{1}{T}\sum_{t=1}^{T} f_i(t), \tag{1}$$

and the variance is

$$\sigma_i = \langle f_i^2(t)\rangle_t - \mu_t^2, \tag{2}$$

where $\langle\rangle_t$ calculates the temporal average over t. Taylor's law of temporal fluctuation scaling reads [4]

$$\sigma_i = a \times \mu_i^b, \tag{3}$$

where a is a positive constant and b is the scaling exponent.

Another form of Taylor's law concerns the ensemble or cross-sectional fluctuation scaling between the variance and the mean of cross-sectional fluctuation signals. For an observation period $t \in (0, T]$ and an observation resolution $\Delta t = 1$ min, we denote $f_i(t)$ the illiquidity time series of stock i over the time interval $[1, T]$, whose mean is

$$\mu(t) = \langle f_i(t)\rangle_i = \frac{1}{N(t)}\sum_{i=1}^{N(t)} f_i(t), \tag{4}$$

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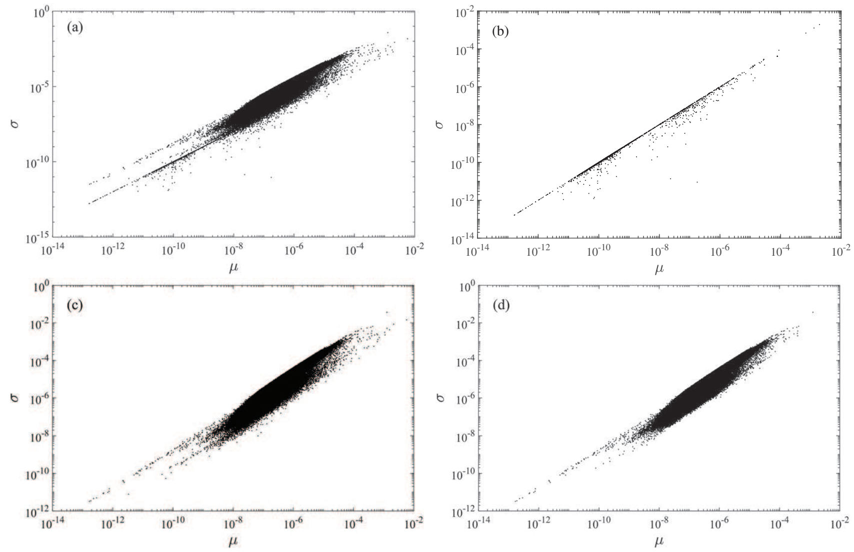

Fig. 1: Taylor's law of cross-sectional fluctuation scaling of the 1 min illiquidity of all the Chinese stocks from 26 July 1999 to 30 December 2011. (a) Full sample. (b) Sample in which each point has only two stocks. (c) Sample in which each point has more than two stocks. (d) Sample in which each point has more than 40 stocks.

where $N(t)$ is the number of stocks available at t, and the variance is

$$\sigma^2(t) = \langle f_i^2(t)\rangle_i - \langle f_i(t)\rangle_i^2. \tag{5}$$

Taylor's law of cross-sectional fluctuation scaling reads [4]

$$\sigma(t) = a \times [\mu(t)]^b, \tag{6}$$

where a is a positive constant and b is the scaling exponent. To estimate the parameters a and b, one can simply rewrite eq. (6) as follows:

$$\ln \sigma(t) = \ln a + b \ln \mu(t), \tag{7}$$

and then performs the ordinary least-squares linear regression.

Taylor's law of fluctuation scaling has been documented in diverse systems [4], such as species abundance in population dynamics [6–8], cosmic rays [9], heavy-ion collisions [10], cell numbers [11], hematogenous organ metastases [12], human single nucleotide polymorphisms [13], traffic fluxes recorded at individual nodes in transportation networks (the number of bytes on Internet, the stream flow in river networks, the number of cars on highways) [14–17], individual health [18], gene expression time series from yeast and human organisms [19], gene network of yeast [20], trading activities in stock markets [21–29], application installations [30], quotation activities and transaction activities in the foreign exchange market [31], and species abundance [32,33], bacterial populations in the human microbiome [33], and tornado outbreaks [34].

Taylor's law can further be extended to higher orders using the k-th *vs.* the j-th cumulants [35], where the special case of $k = 2$ and $j = 1$ recovers the convention expression of Taylor's law. In addition, there is evidence showing that stock illiquidity, a nonadditive quantity, also complies with Taylor's law of temporal fluctuation scaling [36]. Here, we investigate the high-frequency illiquidity time series of Chinese stocks and confirm the presence of Taylor's Law of ensemble fluctuation scaling in stock illiquidity.

Data description. – Our data sets contain the 1 min high-frequency time series of prices $P(t)$ and trading volumes $v(t)$ of 2197 A-share and B-Share stocks traded on the Shenzhen Stock Exchange (SZSE) and the Shanghai Stock Exchange (SHSE) from 26 July 1999 to 30 December 2011. Different stocks have different starting dates.

For each stock i, the 1 min logarithmic returns are calculated from the stock prices $P_i(t)$ as follows:

$$r_i(t) = \ln P_i(t) - \ln P_i(t-1). \tag{8}$$

The 1 min dollar trading volume $v_i(t)$ in the time interval $(t-1, t]$ is computed as the sum over all transactions in the interval. The 1 min illiquidity at time t of stock i is defined as the ratio of the absolute 1 min return to the 1 min trading volume [37]:

$$f_i(t) = |r_i(t)|/v_i(t). \tag{9}$$

Results. – For each minute t, we calculate the mean $\mu(t)$ and the standard deviation $\sigma(t)$ of the cross-sectional illiquidities $f_i(t)$. The points with $\mu = 0$ or $\sigma = 0$ are not included in the following analysis.

Full sample. Figure 1(a) presents the scatter plot of $\sigma(t)$ against $\mu(t)$ at log-log scales. It is found that there is a power-law relationship between $\sigma(t)$ and $\mu(t)$. However, since there are too many data points, we are not able to

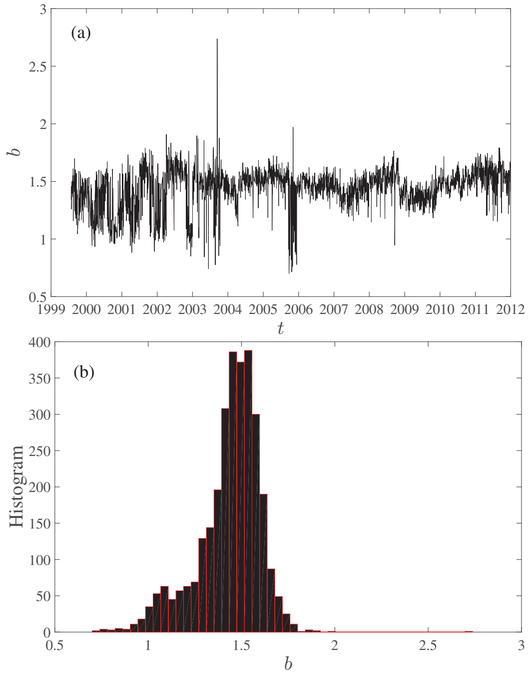

Fig. 2: (Colour online) (a) Evolution of the Taylor exponent b of the 1 min illiquidity time series. (b) Histogram of the Taylor exponents.

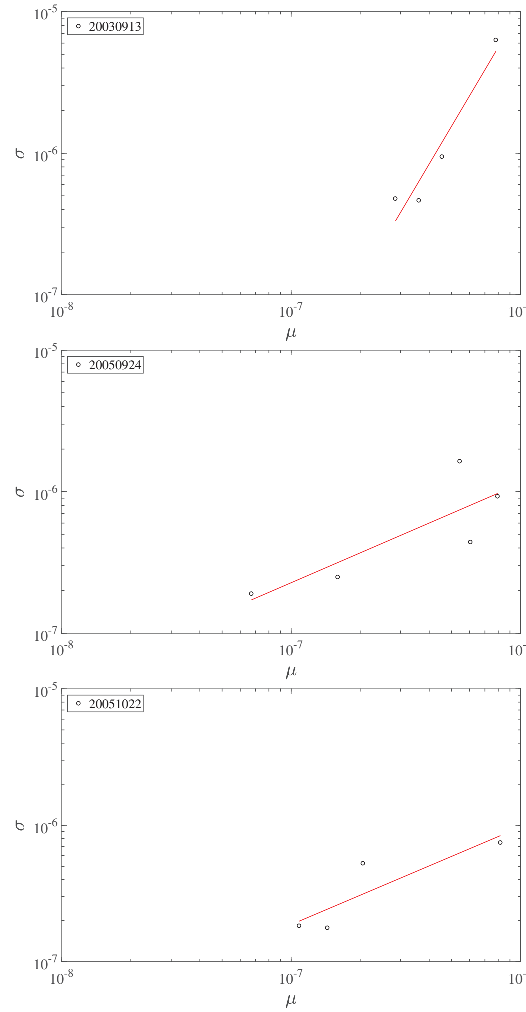

Fig. 3: (Colour online) Dependence of σ on μ for three special days on 2003/09/13, 2005/09/24 and 2005/10/22. The estimated scaling exponents are, respectively, the largest, the smallest and the second smallest among all the daily exponents.

obtain the scaling exponent. We also observe that there are seemingly parallel "lines" in the plot.

A close scrutiny shows that there are many points that are obtained from only two stocks. These points are presented in fig. 1(b), which shows an evident straight line with slope $b \approx 1$ forming an upper boundary. Assume that the 1 min returns of the two stocks at certain time is r_1 and r_2. We have

$$\mu = \frac{r_1 + r_2}{2} \tag{10}$$

and

$$\sigma = \frac{|r_1 - r_2|}{2} = \sqrt{\mu^2 - r_1 r_2}. \tag{11}$$

When $|r_1| \gg |r_2|$ or $|r_1| \ll |r_2|$, we find approximately that $\sigma = \mu$. These points are located on the approximate straight line in fig. 1(b). According to eq. (11), other points lie below the upper boundary $\sigma = \mu$.

In fig. 1(c), we show the sample in which each point has more than two stocks, while in fig. 1(d) we present the sample in which each point has more than 40 stocks. It is observed that the "parallel line" phenomenon becomes weaker and the scaling exponent of the bulk data points seems greater than 1.

Daily evolution. Figure 2(a) shows the daily evolution of the Taylor scaling exponent b of the 1 min illiquidity time series of all Chinese stocks. Each Taylor scaling exponent is obtained from the data in a single day. The exponent fluctuates remarkably in time and exhibits irregular patterns. The mean of b is 1.436 and the standard deviation is 0.169.

Figure 2(b) shows the histogram of the Taylor exponents. We find that b is left-skewed and the skewness is -0.831. In addition, the kurtosis is 5.289, indicating the non-Gaussianity of the distribution of scaling exponents.

We observe that there are many very small exponents and a very large exponent in fig. 2(a). The largest exponent $b = 2.74 \pm 0.53$ was on 2003/09/13. We have $\ln a = 26.35 \pm 7.78$, the p-value of b is 0.0356, the p-value of $\ln a$ is 0.077, and the adjusted R^2 is 0.90. The smallest exponent $b = 0.70 \pm 0.28$ was on 2005/09/24. We have $\ln a = -4.01 \pm 4.22$, the p-value of b is 0.0884, the p-value of $\ln a$ is 0.413, and the adjusted R^2 is 0.57. The second minimum exponent $b = 0.71 \pm 0.28$ was on 2005/10/22. We have $\ln a = -3.98 \pm 4.35$, the p-value of b is 0.1283,

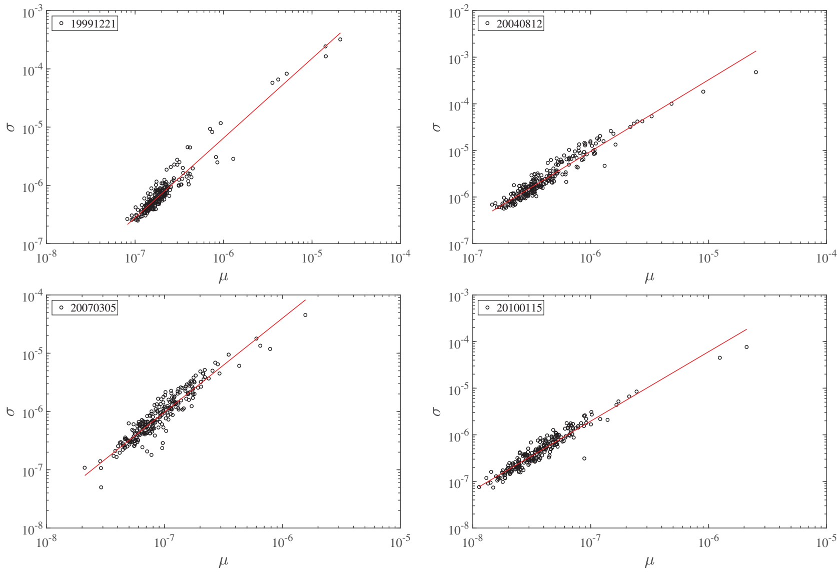

Fig. 4: (Colour online) Taylor's law for four normal days on 1999/12/21, 2004/08/12, 2007/03/05 and 2010/01/15. All the plots exhibit nice power-law scalings. The solid lines are the least-squares fits.

the p-value of $\ln a$ is 0.457, and the adjusted R^2 is 0.64. The third minimum exponent $b = 0.74 \pm 0.10$ was on 2003/06/14. We have $\ln a = -3.23 \pm 1.43$, the p-value of b is 0.0003, the p-value of $\ln a$ is 0.065, and R^2 is 0.89. We find that only the third minimum exponent is significantly different from 0 at the significance level of 0.01. The dependence of σ on μ on the first three days investigated above is illustrated in fig. 3. It is found that all the three plots have very few data points. Further analysis on small Taylor exponents gives similar results.

For comparison, fig. 4 illustrates the dependence of σ on μ for four normal days on 1999/12/21, 2004/08/12, 2007/03/05 and 2010/01/15. There are nice power law scalings over two orders of magnitude in all the four plots, indicating the presence of Taylor's law of cross-sectional fluctuation scaling. For 1999/12/21, we have $b = 1.37 \pm 0.02$, $\ln a = 6.93 \pm 0.37$, the p-value of b is 0.0000, the p-value of $\ln a$ is 0.000, R^2 is 0.93. For 2004/08/12, we have $b = 1.53 \pm 0.03$, $\ln a = 9.65 \pm 0.41$, the p-value of b is 0.0000, the p-value of $\ln a$ is 0.000, R^2 is 0.93. For 2007/03/05, we have $b = 1.61 \pm 0.04$, $\ln a = 12.14 \pm 0.60$, the p-value of b is 0.0000, the p-value of $\ln a$ is 0.000, R^2 is 0.88. For 2010/01/15, we have $b = 1.49 \pm 0.03$, $\ln a = 10.87 \pm 0.50$, the p-value of b is 0.0000, the p-value of $\ln a$ is 0.000, R^2 is 0.92. It is evident that both a and b are significantly different from 0 at the significance level of 0.0001.

According to the above analyses, we discard the days with less than 50 data points. Figure 5 illustrates the evolution of the daily Taylor exponent b and its histogram. The exponent fluctuates in time. The mean and standard deviation of b are, respectively, 1.442 and 0.158. The distribution of b is still left-skewed with a skewness of -0.868. The kurtosis 3.782 becomes smaller than in fig. 2(b).

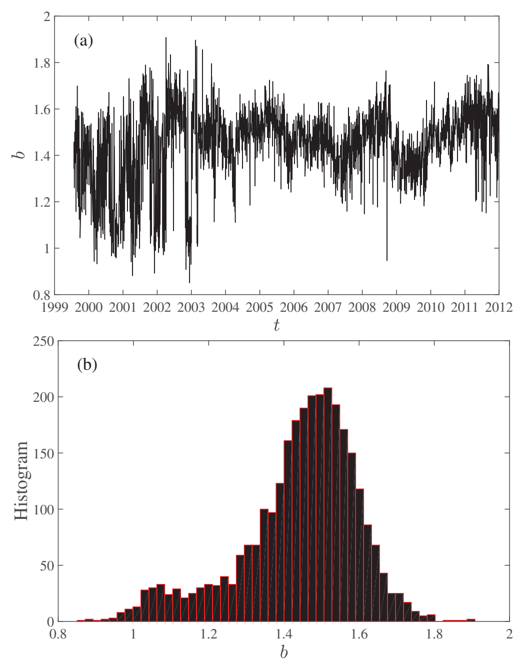

Fig. 5: (Colour online) (a) Evolution of Taylor exponent b in which the days with less than 50 data points have been removed. (b) Histogram of the Taylor exponents.

Relationship to ensemble returns and dispersions. We now check if the Taylor exponent is related to the cross-sectional returns and dispersions. The cross-sectional dispersion is also known as the ensemble variety in econophysics [38–41]. The cross-sectional return $\mu_r(t)$

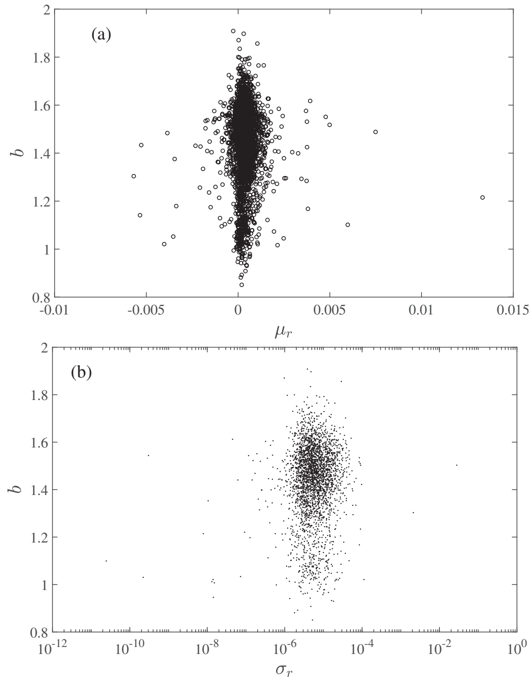

Fig. 6: Scatter plots of b *vs.* μ_r (a) and σ_r (b). No evident dependence is observed.

at time t is defined as follows [38]:

$$\mu_r(t) = \langle r_i(t) \rangle = \frac{1}{N(t)} \sum_{i=1}^{N(t)} r_i(t), \tag{12}$$

where $N(t)$ is the number of stocks included in the calculation. The cross-sectional dispersion $\sigma_r(t)$ at time t is defined as follows [38]:

$$\sigma_r^2(t) = \frac{1}{N(t)} \sum_{i=1}^{N(t)} \left[r_i(t) - \mu_r(t) \right]^2. \tag{13}$$

In fig. 6, we present the scatter plots of b with respect to μ_r and σ_r at the daily level. No evident interdependence is observed. It implies that Taylor's law of cross-sectional fluctuation scaling may not relate to the overall market fluctuations and volatilities.

Summary and discussions. – In this paper, we have performed a cross-sectional analysis of the 1 min high-frequency illiquidity time series of Chinese stocks. The presence of Taylor's law of ensemble fluctuation scaling has been confirmed. We found that the estimated daily Taylor scaling exponent fluctuates around 1.442 in time. We further observed that Taylor's scaling exponents of stock illiquidity are not related to the ensemble mean and ensemble variety of stock returns. Our analysis unveiled a new scaling law of financial illiquidity that complements Taylor's temporal fluctuation scaling law of illiquidity [36]. Our findings also contribute to the literature by documenting Taylor's ensemble fluctuation scaling law on a non-additive quantity.

Our work might stimulate further investigations on Taylor's law to gain a better understanding of financial markets' complex dynamics. One can study financial time series at different sampling frequencies at different stock markets or industrial sectors. Other than trading activities and illiquidity, one can investigate other financial quantities such as bid-ask spread [42], volatility [43], turnover rate [44], and immediate price impact [45–47]. It is interesting as well to check if Taylor's law holds for other financial assets, such as foreign exchange rates, interest rates, equity futures and options, and commodities.

Several models have been developed to explain the emergence of Taylor's law in different systems [14,17,35,48,49]. However, the observed scaling law in stock illiquidity cannot be explained by these models. The microscopic origins of Taylor's law in stock illiquidity remain an open problem. Further research is required.

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