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# Time-dependent Hurst exponent in financial time series

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**Abstract**

We calculate the Hurst exponent $H(t)$ of several time series by dynamical implementation of a recently proposed scaling technique: the detrending moving average (DMA). In order to assess the accuracy of the technique, we calculate the exponent $H(t)$ for artificial series, simulating monofractal Brownian paths, with assigned Hurst exponents $H$. We next calculate the exponent $H(t)$ for the return of high-frequency (tick-by-tick sampled every minute) series of the German market. We find a much more pronounced time-variability in the local scaling exponent of financial series compared to the artificial ones. The DMA algorithm allows the calculation of the exponent $H(t)$, without any a priori assumption on the stochastic process and on the probability distribution function of the random variables, as happens, for example, in the case of the Kitagawa grid and the extended Kalmann filtering methods. The present technique examines the local scaling exponent $H(t)$ around a given instant of time. This is a significant advance with respect to the standard wavelet transform or to the higher-order power spectrum technique, which instead operate on the global properties of the series by Legendre or Fourier transform of $q$th-order moments.


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## 1. Introduction

Modeling and forecasting price return and volatility is a main task of financial research, whose possibility to be successful has been fueled by the demonstration of the persistent behavior of many financial series [1–8]. The long-memory properties of the financial time series entered the econometrics literature in the celebrated autoregressive conditional heteroskedasticity model (ARCH) [3], in its generalization (GARCH) [4] and in the many variants which followed later (see [5] for a review). The ARCH/GARCH models exhibit weak persistent behavior especially on long-horizons, where the time correlation disappears and a simple uncorrelated Itô process is recovered. Stronger long-range correlation is displayed by the fractionally integrated generalized autoregressive conditional heteroskedasticity (FIGARCH) model [6–8]. The main purpose of the present work is to show how to calculate the time-dependent Hurst exponent $H(t)$ using the detrending moving average (DMA) technique. The technique will be applied to artificially generated monofractal series and to two financial series of the German market, a stock market index and a government bond. Our study indicates that the Hurst exponent of financial series shows a much richer time-variability than that of monofractal artificial series.

## 2. Local detrending and scaling of nonstationary stochastic time series

For the sake of clarity, we begin with a summary of the DMA algorithm [9]. The computational procedure is based on calculating the standard deviation about the moving average

$$\sigma_{\mathrm{DMA}} \equiv \sqrt{\frac{1}{N_{\max}-n}\sum_{t=n}^{N_{\max}}[y(t)-\tilde{y}_n(t)]^2}\,, \tag{1}$$

where $\tilde{y}_n(t) \equiv (1/n)\sum_{k=0}^{n-1} y(t-k)$ is the moving average with time window $n$. The function $\sigma_{\mathrm{DMA}}$ is calculated for different values of the moving average window $n$ over the interval $[n, N_{\max}]$, where $N_{\max}$ is the size of the series. The values of $\sigma_{\mathrm{DMA}}$ corresponding to each $\tilde{y}_n(t)$ are plotted as a function of $n$ on log–log axes. The function $\sigma_{\mathrm{DMA}}$ exhibits a power-law dependence with exponent $H$ on $n$ ($\sigma_{\mathrm{DMA}} \propto n^H$). In particular, the exponents $0.0<H<0.5$ and $0.5<H<1.0$ correspond, respectively to negative (anti-persistence) and positive (persistence) correlation, while $H=0.5$ corresponds to an uncorrelated Brownian process.

According to the DFA method [10–12], after dividing the series $y(t)$ in boxes of equal size $n$, a polynomial fit $y_{\mathrm{pol}}(t)$ is calculated in each box. Then, the function

$$\sigma_{\mathrm{DFA}} \equiv \sqrt{\frac{1}{N_{\max}}\sum_{t=1}^{N_{\max}}[y(t)-y_{\mathrm{pol}}(t)]^2}\,, \tag{2}$$

representing a variance about the polynomial fit, is calculated over different size boxes. The relationship $\sigma_{\mathrm{DFA}} \propto n^H$ is obtained for long-memory correlated processes. The DMA algorithm presents higher execution speed and accuracy compared to DFA, due to the better low-pass performance of the moving average compared to the polynomial filter [13].

## 3. Test on artificial series

In this paper, we will use the DMA algorithm to determine the local correlation degree of the series by calculating the local scaling exponent over partially overlapping subsets of the analyzed series. We apply the DMA algorithm on the ensemble of points obtained by the intersection of the signal and a sliding window $W_{\mathrm{s}}$ of size $N_{\mathrm{s}}$, which moves along the series with step $\delta_{\mathrm{s}}$. The scaling exponent is calculated for each subset according to the procedure described above. Thus, a sequence of Hurst exponent values is obtained. The size of this sequence ranges from 1 (in the trivial case of a unique subset coinciding with the entire series) to $N_{\max} - N_{\mathrm{s}}$ (in the case of a sliding window $W_{\mathrm{s}}$ moving point-by-point along the series: i.e., with $\delta_{\mathrm{s}}^{\min} = 1$). The minimum size $N_{\mathrm{s}}^{\min}$ of each subset is defined by the condition that the scaling law $\sigma_{\mathrm{DFA}} \propto n^H$ holds in the subset (typically $N_{\mathrm{s}}^{\min} = 2000$–$3000$). The maximum resolution of the technique is achieved with $N_{\mathrm{s}}^{\min}$ and $\delta_{\mathrm{s}}^{\min}$.

First, we test the feasibility and the accuracy of the dynamic detrending technique on artificial series generated by the random midpoint displacement (RMD) algorithm, which produces signals behaving as fractional Brownian paths with assigned Hurst exponent [14].

## 4. Application to a stock index and a bond index

Next, we apply the DMA algorithm to the log-return of German financial series (tick-by-tick sampled every minute): the DAX (stock index) and the BOBL (government bond). If $p(t)$ indicates the price at the time $t$, the log-return $r(t)$ is $r(t) = \log p(t + \Delta t) - \log p(t)$. Fig. 1 shows the exponent $H(t)$ for (a) an artificial series having $H = 0.5$, (b) the returns of the DAX, and (c) the returns of the BOBL. For the results plotted in Fig. 1, the size of the artificial series is $N_{\max} = 2^{20}$. The size of the subsets is $N_{\mathrm{s}} = 5000$ and the step is $\delta_{\mathrm{s}} = 100$. The parameter $n$ varies from 10 to 1000 with step 2. It is apparent on comparing Fig. 1(a) with Figs. 1(b) and (c) that the artificial series are characterized by a local variability of the correlation exponent weaker than those of the BOBL and DAX series. The small fluctuations exhibited by the $H(t)$ of the artificial series should be considered as the limits of accuracy of the technique. Table 1 shows the means and the standard deviations relative to the average value of the scaling exponents $H(t)$ for the artificial series and for the financial series. The results provide evidence that a more complex evolution dynamics characterizes the financial returns compared to artificial series having the same average value of the Hurst exponent.

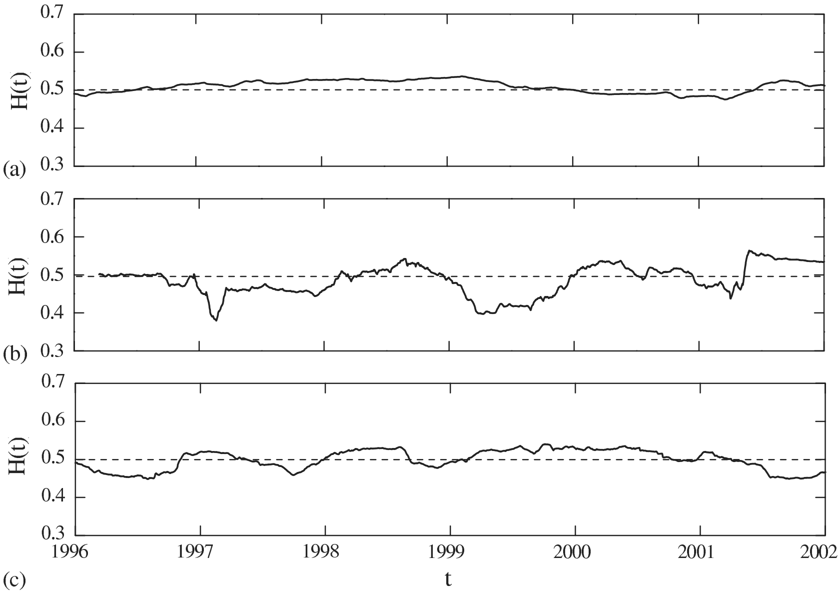


Fig. 1. (a) Time-dependent Hurst exponent $H(t)$ for artificial series with average value $H = 0.5$. (b) Same, but for log-returns of the DAX stock index. (c) Same, but for log-returns of the BOBL bond index.

Table 1
Average value $H$ and standard deviation (relative to the mean) of the Hurst exponent $H(t)$ for the RMD generated artificial series [Fig. 1(a)], for the DAX stock index [Fig. 1(b)], and for the BOBL bond index [Fig. 1(c)]

| | $H$ | $\Delta H(\%)$ |
|---|---|---|
| RMD | 0.5 | 1.47 |
| DAX | 0.490 | 4.16 |
| BOBL | 0.486 | 3.23 |

## 5. Discussion

We have reported preliminary results concerning the local scaling exponent $H(t)$ of financial series and artificial series. We calculated $H(t)$ using the DMA technique [9,15,16]. The ability of the DMA technique to perform such analysis relies on the local scaling properties of function (1). The DMA algorithm allows us to calculate the exponent $H(t)$, without any a priori assumption on the stochastic process and on the probability distribution function of the random variables entering the process, as in the case of the Kitagawa grid and of the extended Kalmann filtering methods [17]. The proposed technique examines the local scaling around a given instant of time. This is a main advance with respect to the standard wavelet transform or the higher-order power spectrum technique, which instead operate on the global properties of the series by Legendre or Fourier transform of $q$th-order moments. Our study indicates several directions for future research. Using the dynamic algorithm here presented, or a variant under development, the multifractal properties of long-range correlated nonstationary series can be analyzed locally rather than globally (as done by the wavelet transform or the higher-order power spectrum technique).

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