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Ballistic deposition with power-law noise: A variant of the Zhang model

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We study a *variant* of the Zhang model [Y.-C. Zhang, J. Phys. (Paris) **51**, 2113 (1990)], ballistic deposition of rods with the length l of the rods being chosen from a power-law distribution $P(l) \sim l^{-1-\mu}$. Unlike in the Zhang model, the site at which each rod is dropped is chosen *randomly*. We confirm that the growth of the rms surface width w with length scale L and time t is described by the scaling relation $w(L,t)=L^{\alpha}w(t/L^{\alpha/\beta})$, and we calculate the values of the surface-roughening exponents α and β. We find evidence supporting the possibility of a critical value $\mu_c \approx 5$ for $d=1$, with $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$ for $\mu>\mu_c$, while for $\mu<\mu_c$, α and β vary smoothly, attaining the values $\alpha=\beta=1$ for $\mu=2$.

It is generally believed that a variety of different models[1–4] of randomly growing rough surfaces can be described in terms of the Kardar-Parisi-Zhang equation[5,6]

$$\frac{\partial h(x,t)}{\partial t}=\nabla^2 h+\tfrac{1}{2}\lambda(\nabla h)^2+\eta(x,t)\,. \tag{1}$$

Here $h(x,t)$ is the height of the surface at time t and position x and $\eta(x,t)$ is a random noise term. One such model is ballistic deposition,[7] for which the surface width seems to follow a scaling form

$$w(L,t)=L^{\alpha}f\left[\frac{t}{L^{\alpha/\beta}}\right]. \tag{2a}$$

The exponents α and β satisfy the general scaling relation[8,9]

$$\alpha+\alpha/\beta=2 \tag{2b}$$

and can be calculated exactly for $d=1$ in the case of normally distributed *uncorrelated* noise $\eta(x,t)$,

$$\alpha=\tfrac{1}{2}\,,\ \beta=\tfrac{1}{3}\,. \tag{2c}$$

Recent experiments on surface growth display exponents quite different from those of (2c): for wetting in a porous medium[10] $\alpha=0.73\pm0.003$ and[11] $\alpha\approx0.81$, $\beta=0.65$; while for growth of bacteria colonies,[12] $\alpha=0.78\pm0.06$.[13] Recently Zhang[14–16] suggested that the anomalous roughening seen in the experiments might arise from noise $\eta(x,t)$ which is *spatially uncorrelated* but with amplitude obeying an algebraic or "power-law" distribution,

$$P(\eta>\xi)=\frac{1}{\xi^{\mu}}\,. \tag{3}$$

He studied this model numerically using a discrete version of Eq. (1), which can be considered also as a version of simultaneous ballistic deposition with switching between even and odd sublattices after each time step.

A mean-field theory,[16,17] which can incorporate a power-law distribution of noise such as (3), predicts the existence of a critical value μ_c, with the property that for $\mu\geq\mu_c$ the large values of η are sufficiently rare that they do not affect α and β, and Eq. (2c) holds. Below $\mu_c=2d+3$ the exponents follow the formulas

$$\alpha=\frac{d+2}{\mu+1},\ \beta=\frac{d+2}{2\mu-d}\,. \tag{4}$$

For $d=1$, (4) predicts a critical value of $\mu_c=5$. However, preliminary calculations on the Zhang model[14–16] and the related problem of directed polymers[18] show no sign of a well defined μ_c.

Here we present simulations on a variant of the Zhang model in which the site at which each rod is dropped is chosen *randomly;* we find values of α and β that are slightly different than those calculated by Zhang and in particular our calculations suggest the existence of a critical value of μ, $\mu_c\simeq5$.

Our model is defined as follows. Consider a $d=1$ lattice of L sites with the height function $h(x,t)$ defined on each site, $1\leq x\leq L$. At $t=0$, $h(x,t)=0$ for all x.

Growth proceeds by the following rules: (i) Choose randomly a position x from $1 \le x \le L$ at which a rod will be dropped. (ii) Choose the length l of the rod using

$$l \equiv [u^{-1/\mu}] , \tag{5}$$

where $0 < u < 1$ is a random variable distributed uniformly and $[u^{-1/\mu}]$ denotes the largest integer number which is less than or equal to $u^{-1/\mu}$. (iii) Attach the rod onto the highest point of the surface at the given location x according to the rule

$$h(x,t+1) \equiv \max[h(x,t), h(x+1,t)-1, h(x-1,t)-1] + l . \tag{6}$$

(iv) The values of h in all other sites remain the same, and periodic boundary conditions are taken at the edges.

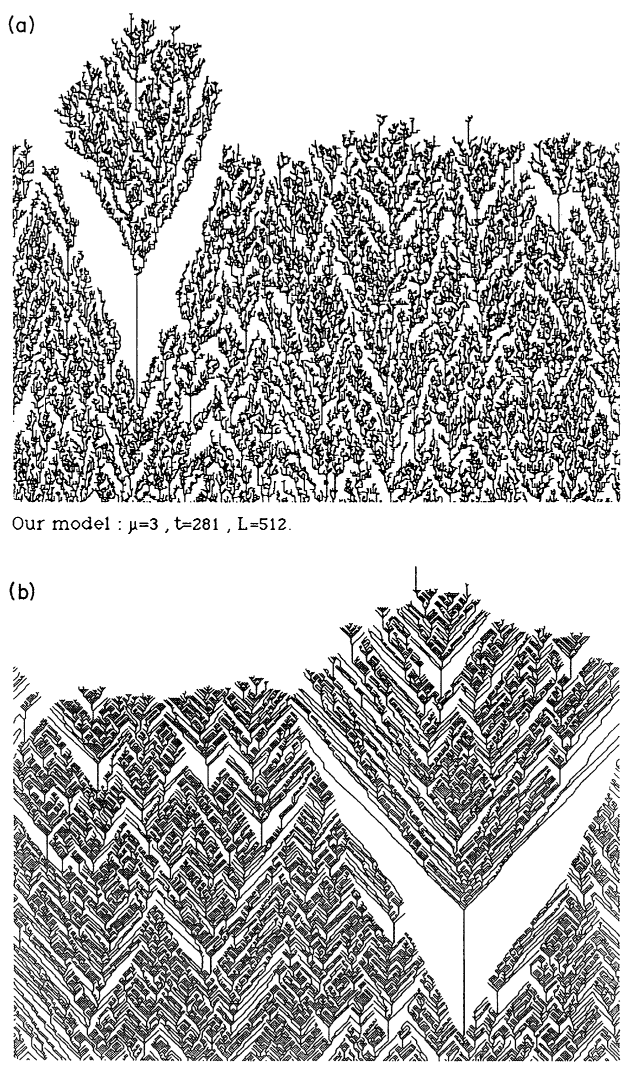

FIG. 1. Anomalous ballistic deposition with power-law noise (3): (a) the present model, and (b) the Zhang model for the case $\mu = 3$ and $L = 512$, at time $t = 281$; the large rod appeared at $t = 206$.

In order to make the time scale identical with that of the Zhang model, we measure the time in Monte Carlo steps per site; i.e., we choose the unit of time to be equal to L so that exactly L rods are deposited during each unit of time.

The underlying role of the rare events is the same for both models: long rods produce big jumps in the surface, which initiate laterally propagating perturbations. However, the surface structure we obtain differs somewhat in visual appearance from those we find using the Zhang model (Fig. 1).

We carried out simulations for a sequence of values of both of the two parameters L and t. Since the results are dominated by rare events, we carried out large numbers N of runs for each pair (L,t).[19]

We analyzed our results by calculating both the rms width defined by[20]

$$w(L,t) \equiv \left\{ \frac{1}{N} \sum_{i=1}^{N} \left[\frac{1}{L} \sum_{x=1}^{L} \left(h_i(x,t) - \frac{1}{L} \sum_{y=1}^{L} h_i(y,t) \right)^2 \right] \right\}^{1/2} , \tag{7a}$$

and the height-height correlation function

$$\mathcal{C}(L,t,\Delta) \equiv \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{L} \sum_{x=1}^{L} [h_i(x,t) - h_i(x+\Delta,t)]^2 \right) \right]^{1/2} \tag{7b}$$

The typical behavior of $w(L,t)$ for fixed L is characterized

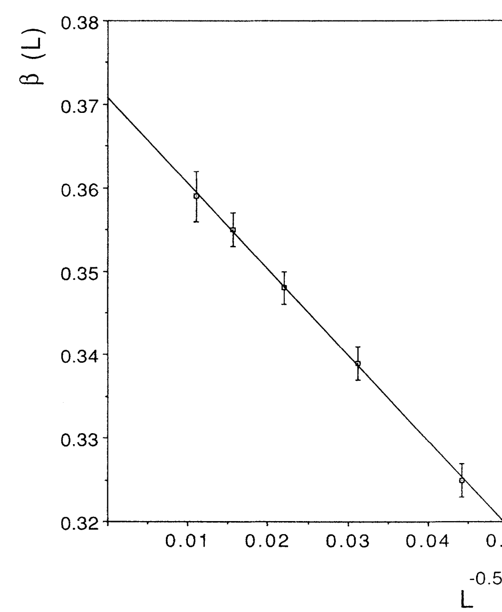

FIG. 2. Estimating β for the case $\mu = 4$. Dependence on the variable $1/L^{\psi}$ of the function $\beta_{\rm eff}(L)$ defined in (8) for the choice $\psi = \frac{1}{2}$; the estimate of $\beta = 0.37$ is given by the intercept at $1/L = 0$. Other choices of ψ fit less well to a straight line but give similar values of β (e.g., $\beta \le 0.38$ if $\psi = \frac{1}{3}$ and $\beta \ge 0.36$ if $\beta = 1$).

by power-law behavior $w \sim t^{\beta}$ for $1 \ll t \ll t_{\times}(L)$ and a time-independent value $w(L,\infty) \sim L^{\alpha}$ for $t \gg t_{\times}(L)$. Here $t_{\times}(L) \sim L^{\alpha/\beta}$ is the crossover time.

To test for the existence of a critical value μ_c, we explored several methods for calculating the anomalous-roughening exponents α and β. We found that to obtain accurate values required judicious extrapolations in the parameters L, t, and Δ.

(a) β. To obtain β, we first calculated effective L-dependent values $\beta_{\mathrm{eff}}(L)$ for each system size L. Since for very small t the behavior is *not* $w \sim t^{\beta}$, it was necessary to average successive slopes of a sequence of time intervals, excluding those near $t=0$ and those in the vicinity of $t \approx t_{\times}$. We found that our successive estimates $\beta_{\mathrm{eff}}(L)$ vary smoothly with L, and could be extrapolated using the formula

$$\beta_{\mathrm{eff}}(L) = \beta - \frac{k_1}{L^{\psi}} . \tag{8}$$

The best fit was found with $\psi \approx \frac{1}{2}$ (see Fig. 2), but the results for β obtained with other values of ψ are within the error bars we quote.[21]

(b) α. To obtain α, we must first obtain accurate estimates of $w(L,\infty)$. We found [Fig. 3(a)] that for a fixed value of L, we could fit $w(L,t)$ for $t \gg t_{\times}$ with the expansion

$$\ln w(L,t) \simeq \ln w(L,\infty) - \frac{k_2}{t^{\psi'}} . \tag{9}$$

The best fit was found with $\psi' \approx 1$, but the results for α

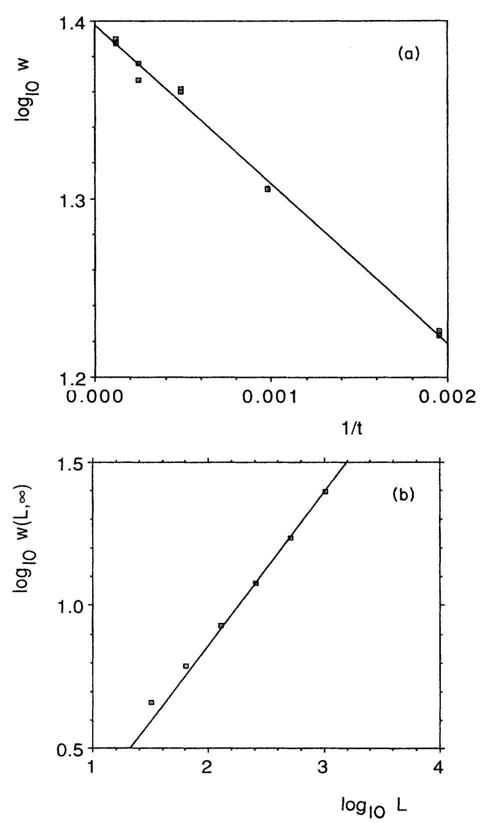

FIG. 3. Estimating α for $\mu=4$. (a) Extrapolation of $w(L,t)$ vs $1/t$ for $L=1024$, $\mu=4$ [cf. Eq. (9)]. (b) Double-log plot showing the $L-$ dependence of extrapolated width $w(L,\infty)$ on L. The slope of the straight line shown corresponds to the extrapolated value of $\alpha=0.54$.

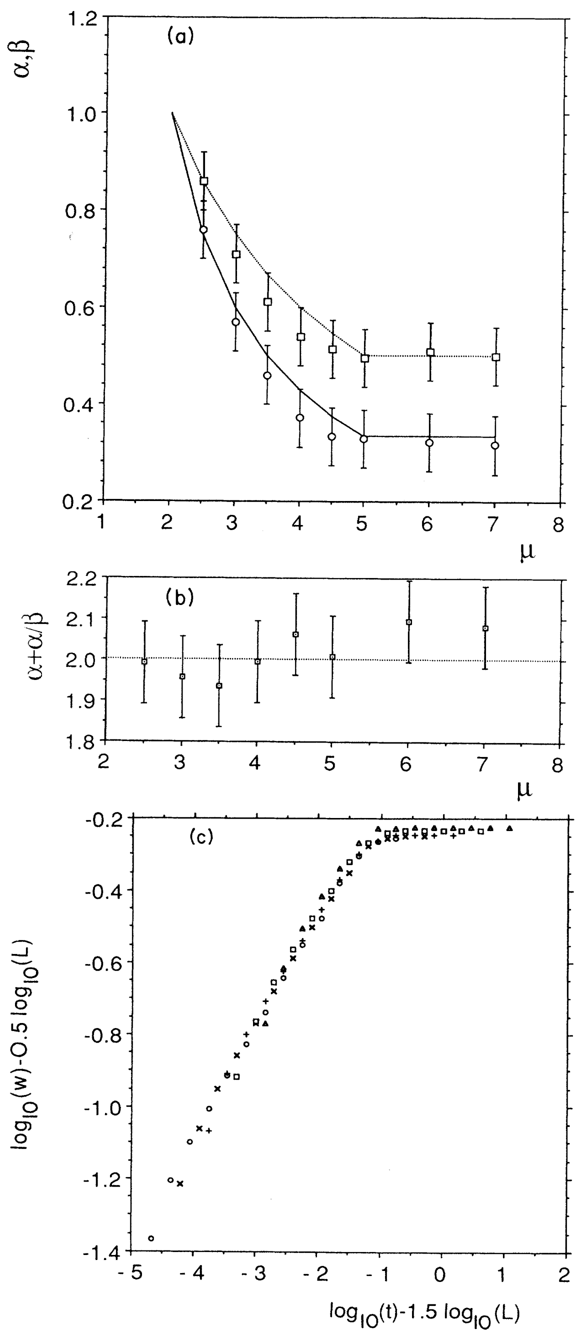

FIG. 4. (a) Comparison of our results on exponents α(□) and β(○) with the prediction (4) of the mean-field theory for α (dotted line) and β (solid line). (b) Check of the scaling relation (2b) relating the exponents α and β. (c) Test of the scaling relation (2a) for $\mu=5$ using the "classical" exponent values, $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$ for different system sizes $L=128$ (△), 256 (□), 512 (+), 1024 (×), and 2048 (○).

obtained with other values of ψ' are within the error bars we quote. We plot [Fig. 3(b)] the sequence of estimates $w(L,\infty)$ obtained from (9) against L and the slopes of successive pairs of points of this plot were then extrapolated against $1/L$ to obtain a value of α appropriate to the $L\rightarrow\infty$ limit.[22]

Figure 4(a) shows the comparison of our results for different values of μ with the mean-field prediction (4). It can be seen clearly that for $\mu \geq \mu_c \simeq 5$ both α and β are almost independent of μ and are very close to $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$. For $\mu<\mu_c$ both exponents deviate from their "classical" values and approach the limiting values $\alpha=1$, $\beta=1$ predicted by the mean-field theory for $\mu=2$. We also confirmed that our numbers are consistent with the scaling relation $\alpha+\alpha/\beta=2$ [Fig. 4(b)], and that for $\mu=5$ $w(L,t)$ obeys (2a) with classical exponent values given by (2c) [Fig. 4(c)].

Thus our results are consistent with the prediction of the mean-field theory (4) that there exists a critical value of $\mu=\mu_c=5.0^{+0.5}_{-1.0}$ above which the anomalous-roughening exponents take on their classical values $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$. It is possible that $\mu_c<5$, but not likely that μ_c is much greater than five as recently claimed by others.[15] The deviations from the mean-field predictions that we find for $2\leq\mu\leq\mu_c$ may be due to inaccuracies in our extrapolation procedures, or to the fact that the theory is only approximate. Even the largest systems studied ($L=2^{13}$) were not large enough to determine the exponents directly, which is the reason that we have used various extrapolation procedures.

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[1]P. Meakin, CRC Crit. Rev. Solid State Mater. Sci. **13**, 143 (1987).

[2]See, e.g., the recent book *Dynamics of Fractal Surfaces,* edited by F. Family and T. Vicsek (World Scientific, Singapore, in press), and references therein.

[3]T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989); *Fractals and Disordered Systems,* edited by A. Bunde and S. Havlin (Springer-Verlag, Berlin, 1991).

[4]Color photographs depicting the rough interface of conventional ballistic deposition appear (courtesy of P. Meakin) in A. Aharony and J. Feder, *Fractals in Physics: Essays in Honor of B. B. Mandelbrot* (Elsevier, Amsterdam, 1990), p. 337, and as Plate 15 (courtesy of F. Family) in D. Stauffer and H. E. Stanley, *From Newton to Mandelbrot: A Primer in Theoretical Physics* (Springer-Verlag, Berlin, 1990). For a striking comparison between experimental results on fractal surfaces and ballistic deposition models, see Plates 18 and 19, respectively, in *Fractal Forms,* edited by E. Guyon and H. E. Stanley (Elsevier, Amsterdam, 1991).

[5]M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[6]The pioneering work on correlations was carried out by E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang, Phys. Rev. A **39**, 3053 (1989).

[7]F. Family and T. Vicsek, J. Phys. A **18**, L75 (1985).

[8]P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, Phys. Rev. A **34**, 5091 (1986).

[9]J. Krug, Phys. Rev. A **36**, 5465 (1987).

[10]M. A. Rubio, C. A. Edwards, A. Dougherty, and J. P. Gollub, Phys. Rev. Lett. **63**, 1685 (1989).

[11]V. K. Horváth, F. Family, and T. Vicsek, J. Phys. A **24**, L25 (1991).

[12]T. Vicsek, M. Cserzö, and V. K. Horváth, Physica A **167**, 315 (1990).

[13]Numerical simulations of surface growth with *spatially correlated* noise yield higher values for α and β; see, e.g., P. Meakin and R. Jullien, Europhys. Lett. **9**, 71 (1989); A. Margolina and M. Warriner, J. Stat. Phys. **60**, 809 (1990); A. Margolina, in *Correlations & Connectivity: Geometric Aspects of Physics, Chemistry & Biology,* edited by H. E. Stanley and N. Ostrowsky, NATO Advanced Study Institutes Ser. E Vol. 188 (Kluwer Academic, Dordrecht, 1990).

[14]Y.-C. Zhang, J. Phys. (Paris) **51**, 2113 (1990).

[15]The Zhang model has attracted considerable interest in the short period since it was proposed. In particular, J.G. Amar and F. Family [J. Phys. A **21**, L79 (1991)] find no evidence for a critical value of μ for $\mu<8$. For results in two dimensions, see R. Bourbonnais, J. Kertész, and D. E. Wolf, J. Phys. II (France) **1**, 493 (1991), and for theoretical arguments, see S. Havlin, S. V. Buldyrev, H. E. Stanley, and G. H. Weiss (unpublished).

[16]Y.-C. Zhang, Physica A **170**, 1 (1990).

[17]J. Krug, J. Phys. I (France) **1**, 9 (1991).

[18]U. M-B. Marconi and Y.-C. Zhang, J. Stat. Phys. **61**, 885 (1990).

[19]To avoid the influence of a cutoff produced by the random number generator (RNG), we used a 48-bit RNG with a period of 2^{48} while the total number of particles in the largest sample did not exceed 2^{36}. Thus the cutoff, $2^{48/\mu}$ has never been reached.

[20]We also studied the general qth moments of the width—which is closely related to the qth moment of the height-height correlation function $\langle|h_i-\langle h\rangle|^q\rangle$ introduced by A.-L. Barabasi and T. Vicsek (unpublished) in connection with their study of multifractality for self-affine structures.

[21]As a consistency check, we also analyzed $\mathcal{C}(L,t,\Delta)$, which displays the *same time dependence as* $w(L,t)$. When we plot the time dependence of the correlation function for $L=1024$ and various values of Δ, we find that slopes of successive pairs of points for each curve allow us to identify the maximum, $\beta_{\max}(\Delta)$. We find, in analogy to (8), that $\beta_{\max}(\Delta)=\beta-k_2/\Delta^{\psi''}$. The best fit was found with $\psi''\approx 1$, but the results for β obtained with other values of ψ'' in the range $[\frac{1}{3},\frac{4}{3}]$ are within the error bars we quote.

[22]We found the same estimates of α by studying $\mathcal{C}(t,L,\Delta)$ for large t and Δ.