Optimal Paths in Disordered Complex Networks

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We study the optimal distance in networks, $\ell_{\rm opt}$, defined as the length of the path minimizing the total weight, in the presence of disorder. Disorder is introduced by assigning random weights to the links or nodes. For strong disorder, where the maximal weight along the path dominates the sum, we find that $\ell_{\rm opt} \sim N^{1/3}$ in both Erdős-Rényi (ER) and Watts-Strogatz (WS) networks. For scale-free (SF) networks, with degree distribution $P(k) \sim k^{-\lambda}$, we find that $\ell_{\rm opt}$ scales as $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda < 4$ and as $N^{1/3}$ for $\lambda \geq 4$. Thus, for these networks, the small-world nature is destroyed. For $2 < \lambda < 3$, our numerical results suggest that $\ell_{\rm opt}$ scales as $\ln^{\lambda-1} N$. We also find numerically that for weak disorder $\ell_{\rm opt} \sim \ln N$ for both the ER and WS models as well as for SF networks.

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Recently much attention has been focused on the topic of complex networks which characterize many biological, social, and communication systems [1–3]. The networks can be visualized by nodes representing individuals, organizations, or computers and by links between them representing their interactions.

The classical model for random networks is the Erdős-Rényi (ER) model [4,5]. An important quantity characterizing networks is the average distance (minimal hopping) $\ell_{\rm min}$ between two nodes in the network of total N nodes. For the Erdős-Rényi network, and the related, more realistic Watts-Strogatz (WS) network [6] $\ell_{\rm min}$ scales as $\ln N$ [7], which leads to the concept of "6 degrees of separation."

In most studies, all links in the network are regarded as identical and thus the relevant parameter for information flow including efficient routing, searching, and transport is $\ell_{\rm min}$. In practice, however, the weights (e.g., the quality or cost) of links are usually not equal, and thus the length of the optimal path minimizing the sum of weights is usually longer than the distance. In many cases, the selection of the path is controlled by the sum of weights (e.g., total cost) and this case corresponds to regular or weak disorder. However, in other cases, for example, when a transmission at a constant high rate is needed (e.g., in broadcasting video records over the Internet) the narrowest band link in the path between the transmitter and receiver controls the rate of transmission. This situation—in which one link controls the selection of the path—is called the strong disorder limit. In this Letter we show that disorder or inhomogeneity in the weight of links may increase the distance dramatically, destroying the "small-world" nature of the networks.

To implement the disorder, we assign a weight or "cost" to each link or node. For example, the weight could be the time τ_i required to transit the link i. The optimal path connecting nodes A and B is the one for which $\sum_i \tau_i$ is a minimum. While in weak disorder all links contribute to the sum, in strong disorder one term dominates it. The strong disorder limit may be naturally realized in the vicinity of the absolute zero temperature if passing through a link is an activation process with a random activation energy ϵ_i and $\tau_i = \exp(\beta\epsilon_i)$, where β is the inverse temperature. Let us assume that the energy spectrum is discrete and that the minimal difference between energy levels is $\Delta\epsilon$. It can be easily shown that if $\beta > \ln 2/\Delta\epsilon$, the value of $\sum_i \tau_i$ is dominated by the largest term, $\tau_{\rm max}$. Thus if we have two different paths characterized by the sums $\sum_i \tau_i$ and $\sum_i \tau'_i$, such that $\tau_{\rm max} > \tau'_{\rm max}$, it follows that $\sum_i \tau_i > \sum_i \tau'_i$ [8].

To generate ER graphs, we start with zN links and for each link randomly select from the total $N(N-1)/2$ possible pairs of nodes a pair that is connected by this link. The WS network [6] is implemented by placing the N nodes on a circle. Initially, each node i is connected with z nodes $i+1, i+2, \ldots, i+z$ and periodic boundaries are implemented. Thus each node has a degree $2z$ and the total number of links is zN. Next we randomly remove a fraction p of the links and use them to connect randomly selected pairs of nodes. When $p = 1$, we obtain a model very similar to the ER graph.

To generate scale-free (SF) graphs, we employ the Molloy-Reed algorithm [9] in which each node is first assigned a random integer k from a power law distribution $P(k > \bar{k}) = (\bar{k}/k_0)^{-\lambda+1}$, where k_0 is the minimal number of links for each node. Next we randomly select a node and try to connect each of its k links with randomly selected k nodes that still have free positions for links.

We expect that the optimal path length in the weak disorder case will not be considerably different from the shortest path, as found for regular lattices [10] and random graphs [11]. Thus we expect that the scaling for the shortest path $\ell_{\rm min} \sim \ln N$ will also be valid for the optimal

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path in weak disorder, but with a different prefactor depending on the details of the graph.

In the case of strong disorder, we present the following theoretical arguments. Cieplak *et al.* [12] showed that finding the optimal path between nodes A and B in the strong disorder limit is equivalent to the following procedure. First, we sort all M links of the network in the descending order of their weights, so that the first link in this list has the largest weight. Since the sum of the weights on any path between nodes A and B is dominated by a single link with the largest weight, the optimal path cannot go through the first link in the list, provided there is a path between A and B which avoids this link. Thus the first link in the list can be eliminated and now our problem is reduced to the problem of finding the minimal path on the network of $M - 1$ links. We can continue to remove links from the top of the list one by one until we pick a link whose removal destroys the connectivity between A and B. This means that all the remaining paths between A and B go through this singly connecting or "red" link [13] and all these paths have the same largest weight corresponding to the red link. To continue optimization among these paths we must select the paths with the minimal second largest term, minimal third largest term, and so on. So we must continue to remove links in the descending order of their weights unless they are red until a single path between A and B, consisting of only red links, remains. Since the assigning of weights to the links is random so is their ordering. Hence the optimization procedure in the strong disorder limit is statistically equivalent to removing the links in random order unless the connectivity between nodes A and B is destroyed.

At the beginning of this process, the chances of losing connectivity by removing a random link are very low, so the process corresponds exactly to diluting the network, which is identical to the percolation model. Only when the concentration of the remaining links approaches the percolation threshold will the chances of removing a singly connected red link [13] become significant, indicating that the optimal path must be on the percolation backbone connecting A and B. Since the network is not embedded in space but has an infinite dimensionality, we expect from percolation theory that loops are not relevant at criticality [14]. Thus, the shortest path must also be the optimal path.

We begin by considering the case of the ER graph that, at criticality, is equivalent to percolation on the Cayley tree or percolation at the upper critical dimension $d_c = 6$. For the ER graph, it is known that the mass of the incipient infinite cluster S scales as $N^{2/3}$ [4]. This result can also be obtained in the framework of percolation theory for $d_c = 6$. Since $S \sim R^{d_f}$ and $N \sim R^d$ (where d_f is the fractal dimension and R the diameter of the cluster), it follows that $S \sim N^{d_f/d}$ so for $d_c = 6$, $d_f = 4$ [15]

$$S \sim N^{2/3}. \tag{1}$$

It is also known [15] that, at criticality, at the upper critical dimension, $S \sim \ell_{\min}^{d_\ell}$ with $d_\ell = 2$, [15] and thus

$$\ell_{\min} \sim \ell_{\mathrm{opt}} \sim S^{1/d_\ell} \sim N^{2/3d_\ell} \sim N^{\nu_{\mathrm{opt}}}, \tag{2}$$

where $\nu_{\mathrm{opt}} = 2/3d_\ell = 1/3$. We expect that the WS model for large N and large p will be in the same universality class as ER.

For SF networks, we can also use the percolation results at criticality. It was found [16] that $d_\ell = 2$ for $\lambda > 4$, $d_\ell = (\lambda - 2)/(\lambda - 3)$ for $3 < \lambda < 4$, $S \sim N^{2/3}$ for $\lambda > 4$, and $S \sim N^{(\lambda-2)/(\lambda-1)}$ for $3 < \lambda \le 4$. Hence, we conclude that

$$\ell_{\min} \sim \ell_{\mathrm{opt}} \sim \begin{cases} N^{1/3} & \lambda > 4, \\ N^{(\lambda-3)/(\lambda-1)} & 3 < \lambda \le 4. \end{cases} \tag{3}$$

To test these theoretical predictions, we perform numerical simulations in the strong disorder limit by randomly removing links (or nodes) for ER, WS, and SF networks and use the Dijkstra algorithm [17] for the weak disorder case. We also perform additional simulations for the case of strong disorder on ER networks using direct optimization [8] and find results identical to the results obtained by randomly removing links [see Fig. 2(a)].

Results for weak disorder for WS graphs with different p are shown in Fig. 1. We propose a scaling formula for ℓ_{opt} similar to the formula derived in [18,19] for the minimal distance on the WS graphs with a different rewiring probability p

$$\ell_{\mathrm{opt}} \sim \frac{n_0(p,z)}{n_1(z)} F\left(\frac{N}{n_0(p,z)}\right), \tag{4}$$

where $n_0(p,z) \sim 1/pz$ is the characteristic graph size at

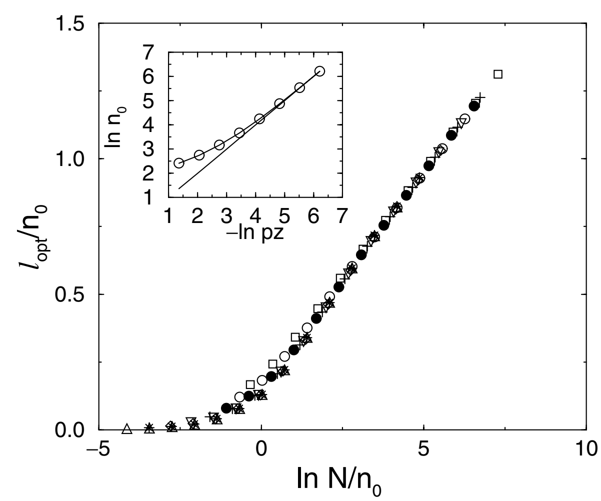

FIG. 1. Scaling plot of ℓ_{opt} on WS graphs for weak disorder as a function $\ln(N/n_0)$ for various values of p and $z = 2$. The inset shows the log-log plot of n_0 versus pz. The different symbols represent different p values: $p = 0.001$ (△), $p = 0.002$ ($*$), $p = 0.004$ (◇), $p = 0.008$ (▽), $p = 0.016$ (+), $p = 0.032$ (●), $p = 0.064$ (○), and $p = 0.128$ (□). Similar results have been obtained for $z = 1$, 4, and 8. Those results scale according to Eq. (4).

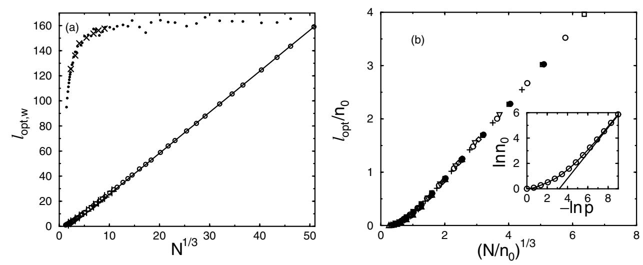

FIG. 2. (a) The dependence of ℓ_{opt} on $N^{1/3}$ for ER graphs for the strong disorder case obtained by direct optimization (+) and by randomly removing links (○). The linear asymptote has a slope of 3.27. Also shown are the successive slopes multiplied by 50 for direct optimization (×) and for randomly removing links (●). (b) Scaling plot of ℓ_{opt} in WS graphs for strong disorder as a function $(N/n_0)^{1/3}$ for various values of p and $z = 2$. The symbols indicating values of p are the same as in Fig. 1. The inset shows a log-log plot of n_0 versus p for $z = 2$.

which the crossover from large- to small-world behavior occurs, $n_1(z) \sim z$ is a correction factor, and $F(x)$ is the scaling function

$$F(x) \sim \begin{cases} \ln x & x \to \infty, \\ x & x \to 0. \end{cases} \tag{5}$$

The scaling variable $x = N/n_0$ indicates the number of nodes with long-range links. As $p \to 0$, this quantity scales as Npz. The quantity $n_0(p, z) \sim 1/pz$ indicates a typical short-range neighborhood of a node with long-range links. We can think of this graph as an ER graph consisting of N/n_0 effective nodes, each representing a typical short-range chainlike neighborhood of size n_0. Thus we conclude that an optimal path connecting any two nodes is proportional to $\ln(N/n_0)$, as in an ER graph, times an average path length through a chain of short-range links. This average path is proportional to the length of this chain n_0 and inversely proportional to the average range $n_1(z)$ of a link in this chain. Ideally $n_1(z) = z$, but in reality it can significantly deviate from z due to finite size effects. Figure 1 shows the scaled optimal path ℓ_{opt}/n_0 versus the scaled variable N/n_0 for $z = 2$ and different values of p. The inset in Fig. 1 shows that $n_0 \sim 1/pz$ as $p \to 0$.

In contrast, for Eq. (4) to be in agreement with Eq. (2) for the strong disorder limit, we have (see Fig. 2)

$$F(x) \sim \begin{cases} x^{1/3} & x \to \infty, \\ x & x \to 0. \end{cases} \tag{6}$$

For large enough z and $p \to 1$, we recover the ER network for which ℓ_{opt} does not depend on z. Thus we can assume $n_0(1, z) = 1$. Using similar scaling arguments as in the case of weak disorder, we assume that as $p \to 0$, $\ell_{opt} \sim z^{-2/3}N$, and hence $n_1(z) \sim z^{2/3}$.

For SF networks, the behavior of the optimal path in the weak disorder limit is shown in Fig. 3 for different degree distribution exponents λ. Here we plot ℓ_{opt} as a function of $\ln N$. All the curves have linear asymptotes, but the slopes depend on λ,

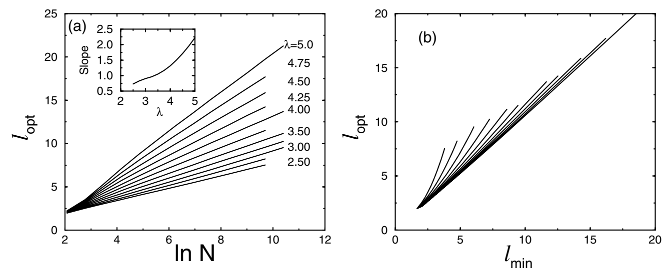

FIG. 3. (a) The dependence of ℓ_{opt} on $\ln N$ for SF graphs in the weak disorder case for various values of λ shown on the graph. The behavior of the asymptotic slope versus λ shown as an inset. (b) The dependence of ℓ_{opt} on ℓ_{min}. The curves from left to right represent increasing values of λ given in (a).

$$\ell_{opt} \sim f(\lambda) \ln N. \tag{7}$$

This result is analogous to the behavior of the shortest path $\ell_{min} \sim \ln N$ for $3 < \lambda < 4$. However, for $2 < \lambda < 3$, ℓ_{min} scales as $\ln\ln N$ [20] while ℓ_{opt} is significantly larger and scales as $\ln N$ [Fig. 3(b)]. Thus weak disorder does not change the universality class of the length of the optimal path except in the case of "ultrasmall" worlds $2 < \lambda < 3$.

In contrast, strong disorder dramatically changes the universality class of the optimal path. Theoretical considerations [Eqs. (2) and (3)] predict that in the case of WS and ER (Fig. 2) and SF graphs with $\lambda > 4$, $\ell_{opt} = N^{1/3}$, while for SF graphs with $3 < \lambda < 4$, $\ell_{opt} \sim N^{(\lambda-3)/(\lambda-1)}$. Figure 4(a) shows the linear behavior of ℓ_{opt} versus $N^{1/3}$ for $\lambda \geq 4$. The quality of the linear fit becomes poor for $\lambda \to 4$. At this value, the logarithmic divergence of the second moment of the degree distribution occurs and one expects logarithmic corrections, i.e., $\ell_{opt} \sim N^{1/3}/\ln N$ [see Fig. 4(b)]. Figure 4(c) shows the asymptotic linear behavior of ℓ_{opt} versus $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda \leq 4$. Theoretically, as $\lambda \to 3$, $\nu_{opt} = (\lambda - 3)/(\lambda - 1) \to 0$, and thus one can expect for $\lambda = 3$ a logarithmic dependence of ℓ_{opt} versus N. Interestingly, for $2 < \lambda < 3$ our numerical results for the strong disorder limit suggest that ℓ_{opt} scales faster than $\ln N$. The numerical results can be fit to $\ell_{opt} \sim (\ln N)^{\lambda-1}$ [see Fig. 4(d)]. Note that the correct asymptotic behavior may be different and this result represents only a crossover regime. We obtain the same results for the SF networks in which the weights are associated with nodes rather then links.

In summary, we study the optimal distance in ER, WS, and SF networks in the presence of strong and weak disorder. We find that in ER and WS networks for strong disorder, the optimal distance ℓ_{opt} scales as $N^{1/3}$. We also study the strong disorder limit in SF networks theoretically and by simulations and find that ℓ_{opt} scales as $N^{1/3}$ for $\lambda > 4$ and as $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda < 4$. Thus, the optimal distance increases dramatically in strong disorder when it is compared to the known small-world result $\ell_{min} \sim \ln N$ and the "small world" nature for these networks is destroyed. Our simulations also suggest that for

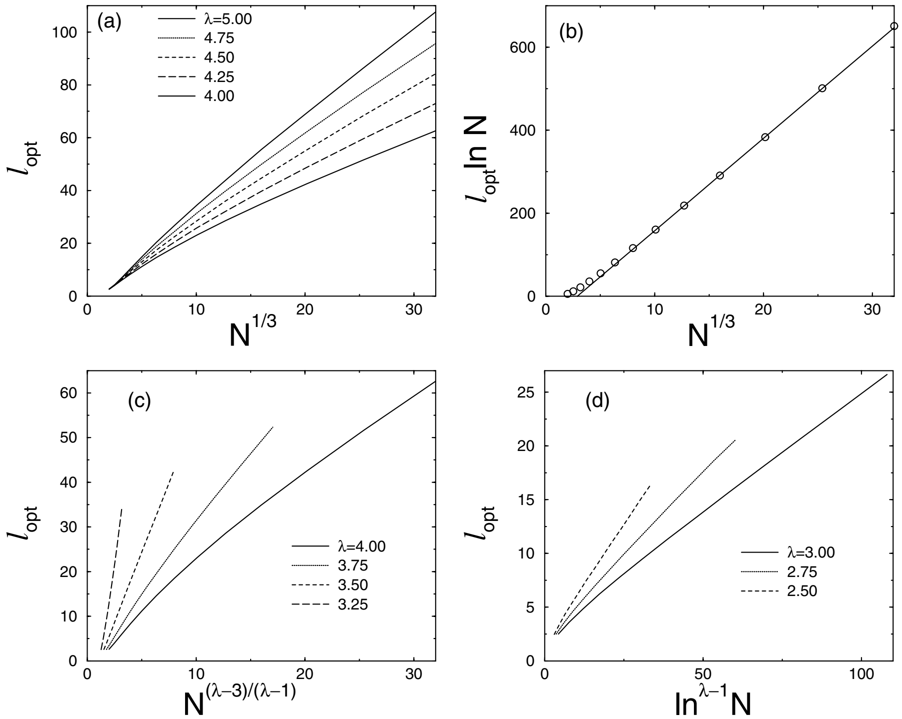

FIG. 4. (a) The dependence of ℓ_{opt} on $N^{1/3}$ for $\lambda \geq 4$. (b) The dependence of $\ell_{opt}\ln N$ on $N^{1/3}$ for $\lambda = 4$. (c) The dependence of ℓ_{opt} on $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda < 4$. (d) The dependence of ℓ_{opt} on $\ln^{\lambda-1}N$ for $\lambda \leq 3$.

$2 < \lambda < 3$, ℓ_{opt} scales as $\ln^{\lambda-1}N$, which is also much faster than the "ultrasmall world" result $\ell_{min} \sim \ln(\ln N)$ [20]. The same scaling as for ℓ_{opt} applies for distances on the minimum spanning tree [21], which behave similarly to paths in strong disorder. We also find numerically that in weak disorder $\ell_{opt} \sim \ln N$ in all types of networks studied.

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