# Images and distributions obtained from affine transformations

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We present a method to obtain the scaling properties for multifractal shapes and distributions obtained from affine transformations. Our analysis defines two scaling exponents $F(\beta)$ and $\tilde{F}(\beta)$; the former relates to geometry, while the latter is associated with the distribution of gradients for networks in Laplacian fields. We determine the *exact* relation between $F(\beta)$ and $\tilde{F}(\beta)$. We argue for electrical networks that more information on scaling behavior may be obtained by only taking into account the maximal current leaving a node. Our picture suggests a new route to characterize the scaling properties of electrical networks in general.

The subject of affine transformations has recently received renewed interest among physicists. For example, in a far-reaching development, Barnsley[1] has shown that an arbitrary image can be "encoded" by a set of affine transformations. This encoding is remarkably efficient in the sense that orders of magnitude fewer "bits" are required to specify the affine transformations than to specify the original image. Apart from a myriad of obvious applications, this development suggests ways in which neural networks may encode information. Affine transformations also arise in other problems of recent interest to physicists, such as dynamical systems.[2]

Our purpose here is to elucidate the *scaling* properties of systems described by affine transformations.[3] We begin by considering a general set of contractive[4] affine transformations of the form

$$W_i(\mathbf{x})\equiv M_i\mathbf{x}+\mathbf{c}_i,\quad i=1,2,\ldots,a \tag{1}$$

where $M_i$ is a matrix independent of $\mathbf{x}$ and $\mathbf{c}_i$ is a translation vector. Without loss of generality we take below $a=2$. The matrices $M_i$ define a tree structure (Fig. 1). At level $n$ there are $2^n$ vertices, which we index by the integers $k=1,2,\ldots,2^n$. The "initial condition" is given by a $d$-dimensional vector $\mathbf{\Delta}$, and at level $n$ each vertex of the tree is associated with a vector $\mathbf{\Delta}_k=(\delta_{k,1},\delta_{k,2},\ldots,\delta_{k,d})$. The length of $\mathbf{\Delta}_k$ is denoted $\Delta_k$.

A fractal image is formed as follows. One starts from the point $\mathbf{x}\equiv(0,0,\ldots,0)$ and iterates by using at every step one of the transformations $W_i$.[1,5] In the limit of infinitely many iterations the points obtained form the fractal image. Consider next the vector $\mathbf{\Delta}\equiv\mathbf{c}_2-\mathbf{c}_1$ as initial conditions for the corresponding matrix tree. For example, consider the system after three iterations. The vector $\mathbf{\Delta}_k=M_1M_2\mathbf{\Delta}$ is the vector between the points $W_1W_2\mathbf{c}_1$ and $W_1W_2\mathbf{c}_2$, where $\mathbf{c}_1=W_1\mathbf{0}$ and $\mathbf{c}_2=W_2\mathbf{0}$. Therefore these two points are covered by a stick of length $\Delta_k$. In general, the approximation to the fractal image created after $n+1$ iterations is covered by the sticks of lengths $\Delta_k$ $(k=1,\ldots,2^n)$. Note that the approximation to the fractal image after $n+1$ iterations are covered by sticks obtained at the $n$th level of the matrix tree.

We next use this covering to extract the scaling properties. The scaling is described in terms of the moments

$$Z(\beta)\equiv\sum_{k=1}^{2^n}(\Delta_k)^\beta\ . \tag{2a}$$

This defines a hierarchy of exponents $F(\beta)$,

$$Z(\beta)\propto 2^{-nF(\beta)}\ . \tag{2b}$$

In general, the scaling behavior can be found from the partition function[6] $\Gamma(q,\tau)\equiv\sum_k p_k^q/l_k^\tau$, where $p_k$ is the fraction of points covered by $l_k$, and where $\tau(q)$ is defined by $\Gamma[q,\tau(q)]=1$. For our covering we have $l_k=\Delta_k$. Since every stick contains two points, all $p_k$ are equal, $p_k=2^{-n}$. Thus we have the correspondence $q\leftrightarrow -F(\beta)$ and $\tau(q)\leftrightarrow-\beta$. In particular, the value of $\beta$ at which $F(\beta)=0$ is the *Hausdorff dimension D* of the fractal image. Here $D$ is defined through $N\propto\ell^{-D}$, where $N$ is the total number of boxes of edge $\ell$ in a covering. If $l_k=\ell$ and $q=0$, then $\Gamma(0,\tau(0))=\sum_{k=1}^N\ell^{-\tau(0)}=N\ell^{-\tau(0)}$ is set equal to 1. Hence $D=-\tau(0)$ and $F(\beta)=0$ for $\beta=D$.

The framework developed above can be used for describing not only the *geometrical* scaling properties of a

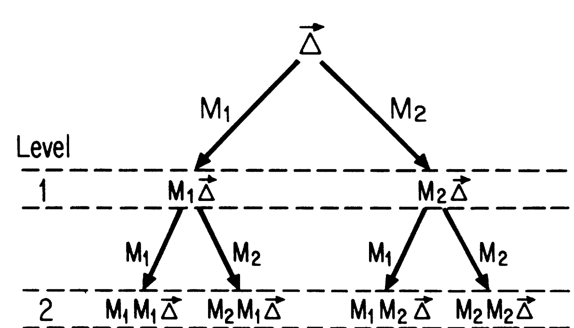


FIG. 1. Tree structure from which all scaling properties can be derived for self-affine systems. $M_k$ $(k=1,\ldots,a)$ are matrices mapping vectors at one level to a set of vectors at the next level. Here $a$ is taken to be 2.

 

fractal object, but also the *dynamical* scaling properties when placing a connected fractal set in a Laplacian field. The corresponding distribution of gradients can have moments that exhibit a whole hierarchy of exponents, even when the geometry of the fractal set is self-similar. To be specific, suppose we consider a self-similar fractal network that consists of $a$ copies of itself. For this we solve the Laplace equation with boundary conditions given by the potentials at a number $d+1$ of nodes, or equivalently, by $d$ gradients (voltage drops) between these nodes. From these gradients, the corresponding $d$ gradients in the $i$th copy ($i=1,2,\ldots,a$) can be obtained through a $d$-dimensional matrix $M_i$ (since the Laplace equation is linear).[7] The scaling properties of the distribution of gradients are determined by the matrix tree defined by the $M_i$'s. Based on the considerations in the previous paragraph we notice that the distribution of gradients can be visualized geometrically, choosing a set of translational vectors $\mathbf{c}_i$ (Fig. 2).

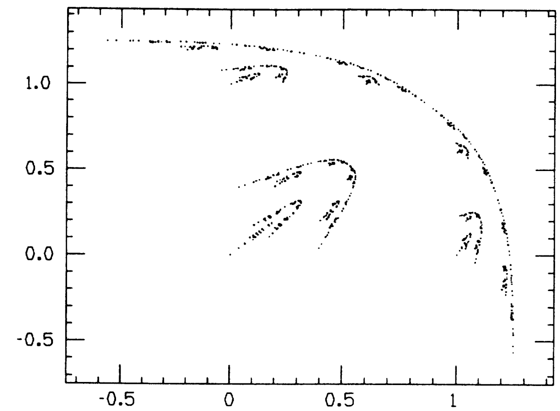

FIG. 2. Geometrical visualization of the voltage-drop distribution in the Sierpinski-gasket resistor network. The point (0,0) is mapped by iterative use of three affine transformations $W_i(\mathbf{x})\equiv M_i\mathbf{x}+\mathbf{c}_i$ ($i=1,2,3$) as explained in the text. The matrices are (Ref. 7) $M_1=\frac{1}{5}\begin{pmatrix} 1 & 0 \\ -1 & 3\end{pmatrix}$, $M_2=\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & 2\end{pmatrix}$, and $M_3=\frac{1}{5}\begin{pmatrix} 3 & -1 \\ 0 & 1\end{pmatrix}$. The translational vectors was taken to be $\mathbf{c}_1=(1,0)$, $\mathbf{c}_2=(0,0)$, and $\mathbf{c}_3=(0,1)$.

Although the scaling structure for the distribution of gradients from the formulation above is naturally given by $F(\beta)$,[8] the quantities usually considered[9] are the moments

$$\tilde{Z}(\beta)\equiv \sum_{k=1}^{2^n}\sum_{l=1}^{d}|\delta_{k,\ell}|^\beta , \tag{3a}$$

based on the components $\delta_{k,\ell}$ ($a$ is taken to be 2). Equation (3a) defines a set of exponents $\tilde{F}(\beta)$,

$$\tilde{Z}(\beta)\propto 2^{-n\tilde{F}(\beta)} . \tag{3b}$$

One observes that while $F(\beta)$ is *independent* of initial conditions, this is in general not true for $\tilde{F}(\beta)$ when $\beta$ is negative. Some of the matrices will usually have negative elements, and zero components can therefore occur for particular choices of initial vector $\Delta$. The number of such "critical" vectors $\Delta$ which lead to vectors $\Delta_k$ with a zero component will typically *increase* with the level of construction.

One might think that if the critical vectors $\Delta$ are not considered, then the full scaling structure can be obtained from $\tilde{F}(\beta)$. We shall show below that this is not so. For this purpose, we find the relation between $F(\beta)$ and $\tilde{F}(\beta)$. We begin by associating with each path j through the matrix tree[10] (Fig. 1) an *energy*[11] $E_{\mathrm{j}}$, defined by

$$\Delta_{k(\mathrm{j})}\propto a^{-nE_{\mathrm{j}}} . \tag{4}$$

Let further $\mathcal{N}(E)dE$ be the number of energies $E_{\mathrm{j}}$ between $E$ and $E+dE$. In the limit $n\to\infty$, this defines an *entropy* $S(E)$ by the relation

$$\mathcal{N}(E)\propto a^{nS(E)} . \tag{5}$$

From (4) and (5), the partition function (2a) can be expressed as an integral on $E$. The integrand $a^{-n[\beta E-S(E)]}$ is maximal for that value of $E$ satisfying

$$\beta=S'(E) . \tag{6a}$$

Hence $F(\beta)$ is the Legendre transform of $S(E)$,

$$F(\beta)=\beta E-S(E) . \tag{6b}$$

In order to relate $F(\beta)$ and $\tilde{F}(\beta)$, we study separately the two cases $\beta>0$ and $\beta<0$; for $\beta=0$, (3a) and (3b) imply that $\tilde{F}(0)=F(0)=-1$. For the sake of simplicity we take below $d=2$ We emphasize, however, that our results extend to any higher dimension. For $\beta>0$,

$$\begin{aligned}|\delta_{k,1}|^\beta+|\delta_{k,2}|^\beta&=(|\delta_{k,1}|^2)^{\beta/2}+(|\delta_{k,2}|^2)^{\beta/2}\\ &\le(|\delta_{k,1}|^2+|\delta_{k,2}|^2)^{\beta/2}\\ &\quad+(|\delta_{k,1}|^2+|\delta_{k,2}|^2)^{\beta/2}\\ &=2(|\delta_{k,1}|^2+|\delta_{k,2}|^2)^{\beta/2} ,\end{aligned} \tag{7}$$

which by (3a) and (3b) yields

$$\tilde{Z}(\beta)\le 2Z(\beta) . \tag{8}$$

To obtain a second bound, we note that

$$\begin{aligned}(|\delta_{k,1}|^2+|\delta_{k,2}|^2)^{\beta/2}&=[(|\delta_{k,1}|^\beta)^{2/\beta}+(|\delta_{k,2}|^\beta)^{2/\beta}]^{\beta/2}\\ &\le[(|\delta_{k,1}|^\beta+|\delta_{k,2}|^\beta)^{2/\beta}+(|\delta_{k,1}|^\beta+|\delta_{k,2}|^\beta)^{2/\beta}]^{\beta/2}\\ &=[2(|\delta_{k,1}|^\beta+|\delta_{k,2}|^\beta)^{2/\beta}]^{\beta/2}=2^{\beta/2}(|\delta_{k,1}|^\beta+|\delta_{k,2}|^\beta) .\end{aligned} \tag{9}$$

Hence

$$Z(\beta)\le 2^{\beta/2}\tilde{Z}(\beta) . \tag{10}$$

Since the bounds of (8) and (10) differ by a constant, the exponents are exactly the same,

$$F(\beta)=\tilde{F}(\beta), \quad \beta\geq 0 \ . \tag{11}$$

For $\beta<0$ we first note that

$$[\min(|\delta_{k,1}|,|\delta_{k,2}|)]^{\beta}=\max(|\delta_{k,1}|^{\beta},|\delta_{k,2}|^{\beta})\leq|\delta_{k,1}|^{\beta}+|\delta_{k,2}|^{\beta}$$
$$\leq 2[\max(|\delta_{k,1}|^{\beta},|\delta_{k,2}|^{\beta})]=2[\min(|\delta_{k,1}|,|\delta_{k,2}|)]^{\beta} \ , \tag{12}$$

so that $\tilde{F}(\beta)$ can be obtained using the *smaller* of the components $\delta_{k,1}$ and $\delta_{k,2}$. Next, we find the Legendre transform $\tilde{S}(E)$ of $\tilde{F}(\beta)$. This is given by the number $\tilde{\mathcal{N}}(E)dE\propto a^{n\tilde{S}(E)}dE$ of energies $\tilde{E}_{\rm j}$ between $E$ and $E+dE$, where $\tilde{E}_{\rm j}$ is defined by the behavior of the smaller vector components,

$$\min(|\delta_{k,1}|,|\delta_{k,2}|)\propto a^{-N\tilde{E}_{\rm j}} \ . \tag{13}$$

From (4) and (13) we notice that if the phase $\theta$ of $(\delta_{k,1},\delta_{k,2})$ is sufficiently close to an integer times $\tau/2$, then $\tilde{E}_{\rm j}$ can be large even when $E_{\rm j}$ is small. Thus $\tilde{\mathcal{N}}(E)dE$ not only includes the number of energies $E_{\rm j}$ between $E$ and $E+dE$, but also the number of energies $E_{\rm j}$ below $E$ for which the associated vectors $\nabla_{k({\rm j})}$ have a value of $\tilde{E}_{\rm j}$ between $E$ and $E+dE$. To be more concrete, let $g_{\hat{E}}(\theta)d\theta\,\mathcal{N}(\hat{E})d\hat{E}$ denote the number of vectors with phase between $\theta$ and $\theta+d\theta$ and energy $E_{\rm j}$ between $\hat{E}$ and $\hat{E}+d\hat{E}$. Then the number $\mathcal{I}(\hat{E},E)d\hat{E}\,dE$ of vectors $(\delta_{k,1},\delta_{k,2})$ associated with an energy $E_{\rm j}$ in the range $(\hat{E},\hat{E}+d\hat{E})$, and $\tilde{E}_{\rm j}$ in the range $(E,E+dE)$ is (Fig. 3)

$$\mathcal{I}(\hat{E},E)d\hat{E}\,dE=\sum_{p=0}^{3} g_{\hat{E}}(p\pi/2)\delta\theta(\hat{E}-E)\mathcal{N}(\hat{E})d\hat{E}\,dE \ , \tag{14a}$$

where

$$\delta\theta(\hat{E}-E)=2a^{n(\hat{E}-E)} \ . \tag{14b}$$

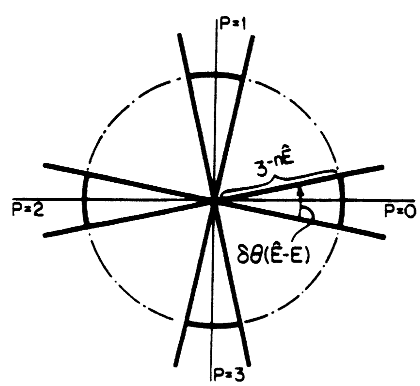


FIG. 3. Contributions to $\tilde{S}(E)$ from $\hat{E}$ shown schematically. The tips of the vectors $(\delta_{k,1},\delta_{k,2})$ with associated energy $E_{\rm j}=\hat{E}$ have a phase distribution $g_{\hat{E}}(\theta)$ along the circle shown. However, the vectors with tips on the solid sections of the circle have correspondingly $\tilde{E}_{\rm j}=E$. The number of these vectors is given by the integrand in (14a).

Now, $\tilde{\mathcal{N}}(E)=\int_{\hat{E}\leq E}\mathcal{I}(\hat{E},E)d\hat{E}$. Substituting (5) and (14b) in (14a) yields an integrand containing the factor $a^{n[S(\hat{E})+\hat{E}-E]}$. To evaluate $\tilde{\mathcal{N}}(E)$ and thereby find $\tilde{S}(E)$ we use steepest descent, distinguishing carefully the two cases $E>E_c$ and $E\leq E_c$, where $E_c$ denotes the energy at which $S(E)+E$ is maximal, i.e., at which $S'(E)+1=0$. Assuming that $g_E(\theta)$ is well defined, nonzero, and bounded at $p\pi/2$, we have

$$\tilde{S}(E)=\begin{cases} S(E_c)+E_c-E & \text{if } E>E_c \\ S(E) & \text{if } E\leq E_c \ . \end{cases} \tag{15}$$

Equation (15) provides a relation between $\tilde{S}$ and $S$. Finally, we obtain a relation between $\tilde{F}$ and $F$, using (6). (i) For $E\leq E_c$, $\tilde{S}(E)=S(E)$, so the free energies $\tilde{F}(\beta)$ and $F(\beta)$ are the same for every $\beta\geq\beta_c\equiv S'(E_c)\equiv-1$. (ii) For $E>E_c$, $\tilde{S}'(E)=-1$, so the Legendre transform of $\tilde{S}(E)$ is just a constant $\tilde{F}(\beta_c)$; hence all the scaling properties given by $S(E)$ above $E=E_c$ are lost. A *phase transition* occurs at $\beta_c$ where the *dominating* terms in the partition function $\tilde{Z}(\beta)$ change abruptly. Our analysis shows that $\tilde{F}(\beta)$ and $F(\beta)$ separate at $\beta=\beta_c=-1$, below which $\tilde{F}(\beta)$ is linear with the slope $E_{\max}$ equal to the upper bound on the energies. The phase transition also predicts a finite entropy[12]

$$\tilde{S}(E_{\max})=S(E_c)+E_c-E_{\max}=|F(-1)|-E_{\max} \tag{16}$$

as $\beta\to-\infty$.

Analogous to the result above that $\tilde{F}(\beta)$ can be found using the smaller components, we notice that $F(\beta)$ can be obtained using the *larger* components. This follows since

$$[\max(|\delta_{k,1}|,|\delta_{k,2}|)]^2\leq|\delta_{k,1}|^2+|\delta_{k,2}|^2$$
$$\leq 2[\max(|\delta_{k,1}|,|\delta_{k,2}|)]^2 \ . \tag{17}$$

Then, by raising all expressions to the power $\beta/2$, we have for $\beta$ positive

$$[\max(|\delta_{k,1}|,|\delta_{k,2}|)]^{\beta}\leq\Delta_k^{\beta}\leq 2^{\beta/2}[\max(|\delta_{k,1}|,|\delta_{k,2}|)]^{\beta} \ . \tag{18}$$

For $\beta$ negative the inequalities are reversed. This result gives a strong suggestion that for general networks more information on the scaling behavior may be obtained by only taking the maximal current leaving (or entering) a node into account.

In conclusion, we have developed a new framework from which the scaling properties of systems described by affine transformations can be found. In particular, we have presented a novel view for characterizing networks in Laplacian fields. For a class of networks, we find that the "sensitivity to initial conditions" for $\beta<0$ and the phase transition at $\beta=-1$ are *universal* results. In order to avoid the problem of scaling for negative moments of the gradient distribution, our picture suggests in general that only the maximal gradient at each node should be considered. However, to obtain full information of the scaling structure, one must perceive the underlying set of affine transformations.

How broad is the universality class described by $\beta_c=-1$? As shown here, it certainly contains every network whose gradient distribution is described by a finite number of (nontrivial) affine transformations. Random networks, however, do not meet this restriction. Nevertheless, for random networks approximations to the gradient distribution may be found similar to Barnsley's approximations to chaotic fractals. How well such approximations describe the scaling properties remains to be shown in both cases.

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[1] M. F. Barnsley and A. D. Sloan, BYTE **13**, 215 (1988); M. F. Barnsley, *Fractals Everywhere* (Academic, Orlando, 1988).

[2] A general example is baker's transformation [see e.g., H. G. Schuster, *Deterministic Chaos* (Physik-Verlag, Weinheim, 1984)], or the pruning-front transformation [P. Cvitanović, G. Gunaratne, and I. Procaccia, Phys. Rev. A **38**, 1503 (1988)].

[3] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982). Recent work on affine transformations of direct relevance to topics in physics research is described in T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).

[4] A contractive transformation moves points closer together.

[5] In the description in Ref. 1 the affine transformations $W_i$ are chosen stochastically with probabilities $p_i$. However, for the sake of simplicity, the deterministic process described here corresponds to the case where all these are equal.

[6] See, e.g., T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986); H. E. Stanley and P. Meakin, Nature **335**, 405 (1988).

[7] P. Alstrøm, D. Stassinopoulos, and H. E. Stanley, Physica A **153**, 20 (1988).

[8] For example, $F(2)$ is related to the spectral dimension $d_s$, $d_s=2/[F(2)+1]$ (Ref. 7).

[9] R. Rammal, C. Tannous, P. Breton, and A. M. S. Tremblay, Phys. Rev. Lett. **54**, 1718 (1985); L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **31**, 4725 (1985); R. Blumenfeld and A. Aharony, J. Phys. A **18**, L443 (1985).

[10] Every path (Fig. 1) is in 1:1 correspondence with a vector $\mathbf{j}=(j_0,j_1,\ldots)$, $j_l$ is 0 (left) or 1 (right). At level $n$, the vertices are labeled by $k=1+\sum_{m=0}^{n-1} j_m 2^m$.

[11] For a general introduction to the thermodynamical formalism, see, e.g., D. Ruelle, in *Encyclopedia of Mathematics and its Applications,* edited by G. -C. Rota (Addison-Wesley, MA, 1978); Vol. 5; M. J. Feigenbaum, J. Stat. Phys. **46**, 919 (1987); **46**, 925 (1987); T. Bohr and D. Rand, Physica D **25**, 387 (1987).

[12] The finite entropy at $E_{\max}$ has been observed for the Sierpinski gasket [S. Roux and C. D. Mitescu, Phys. Rev. B **35**, 898 (1987)], and the value found to be in accordance with (16) (Ref. 7).