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Fractal dimension of the accessible perimeter of diffusion-limited aggregation

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We directly measure the fractal dimension D_A of the accessible perimeter for diffusion-limited aggregation. We find $D_A = D$, where D is the fractal dimension. Moreover, the number of inaccessible sites also scales as the mass of the entire aggregate, stabilizing at 36–37% of the total perimeter after a short transient. Hence, the well-known anomalies in the multifractal analysis of the growth probability distribution are not a result of the possibility $D_A < D$.

Since its introduction in 1981 as a computer algorithm,[1] diffusion-limited aggregation has been used to describe the common features of a vast set of seemingly unrelated phenomena (such as electrodeposition, dendritic growth, and dielectric breakdown). A great deal of work has been devoted to understanding its theoretical properties and fitting them into a coherent framework, but progress has been slow and complete success has not been achieved.

The algorithm itself is very simple. It consists of releasing particles one at a time from far away and letting them execute a random work until they first touch a cluster site, stick to it permanently, and become part of the growing cluster.[1,2] It generates highly ramified, tenuous, low-density figures that have been found to closely match a range of experimental observed structures[3-7] (see Fig. 1). The empty areas never get filled in, due to the strong screening effects exerted by the outer branches of the structure on the inner parts. The most interesting property of diffusion-limited aggregation (DLA) clusters is self-similarity: The aggregates are fractals, whose total mass M scales with the radius of gyration R as

$$M \sim R^D , \quad (1)$$

and D, the fractal dimension, is roughly 1.7 in two dimensions. A great deal is known about the fractal and multifractal properties of DLA clusters.[8]

Another interesting aspect is the growth site probability distribution $\{p_i\}_{i \in S}$. Here p_i in the probability that site i belonging to the perimeter S of the aggregate will grow at the next time step. This distribution has been found to have multifractal behavior. The exponent $\tau(q)$ defined by[9,10]

$$\sum_{i \in S} p_i^q \sim R^{-\tau(q)} \quad (2)$$

does not depend on q in a linear fashion. The sum in (2) runs over all of the perimeter sites i.

The multifractal property is not easily delineated, since the strong screening effect mentioned before implies that the sites at the perimeter of the inner branches have a very small probability of growing. Unfortunately, the small probabilities are crucial in determining multifractality, and are the most difficult to measure with the necessary accuracy.

If $q \rightarrow 0$ in (2), all the growth sites with $p_i \neq 0$ will have the same weight; in other words, $-\tau(0)$ can be viewed as the fractal dimension of the "active" portion of the perimeter of the cluster, which we will call the *accessible perimeter*. In analogy with (1) we write

$$M_A \sim R^{D_A} , \quad (3)$$

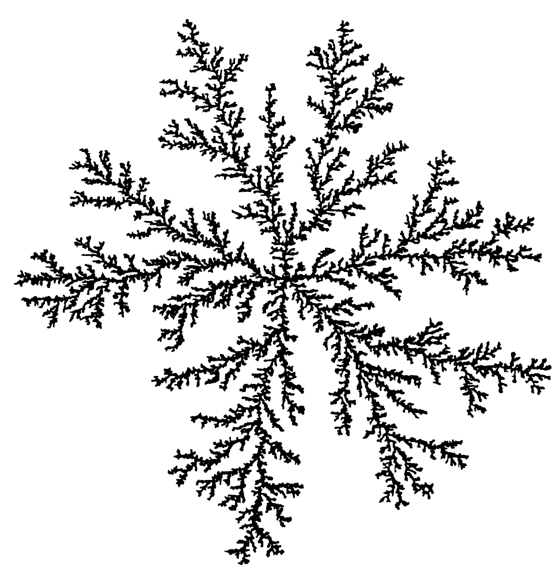

FIG. 1. A DLA cluster studied in this work, with 100 000 particles.

where M_A is the number of perimeter sites *that have growth probability different from zero;* we may write, formally,

$$M_A = \sum_{\substack{i \in S \\ p_i \neq 0}} 1 .$$

In a practical simulation, an apparent growth probability p_i of zero might be a consequence of numerical inadequacy or it might be a real effect (the simplest example being a perimeter site enclosed by the cluster). Three groups[11-13] have measured the multifractal exponents for values of q that include $q=0$. Although using very different methods, all quote values of

$$\tau(0) = -D_A \tag{4}$$

significantly different from the fractal dimension of the cluster, D ($\cong 1.7$). Moreover, their estimates for D_A do not agree, ranging from 1.4 to 1.62.[11-13] Therefore, it is of interest to measure the perimeter exponent independently and check the possibility that the accessible perimeter exponent D_A could be different from the bulk exponent D.

The simulation is carried out as follows. First, we grow the cluster up to M particles. Then the perimeter of the cluster is marked and partitioned into accessible and inaccessible sites. An accessible site is one that, in principle, can be reached by a random walk coming from infinity.[14] Therefore, an accessible site is a perimeter site connected by at least one "on-lattice path" to an open area. Drawing a box that includes the entire cluster without touching it, and "coloring in" from the boundary of the box, i.e., marking in some way every site in the box directly connected to its boundary through a path that did not step on the cluster or on another perimeter site, we find the *accessible* perimeter as intersection of the perimeter of the cluster with the "colored" sites. The complement of the accessible perimeter to the total perimeter is then the *inaccessible* perimeter, and typically it is composed of perimeter sites sitting at a corner of the cluster [Fig. 2(a)], perimeter sites obstructed by other perimeter sites [Fig. 2(b)], and perimeter sites enclosed by the cluster [Fig. 2(c)].

Measurements of the perimeter, its accessible and inaccessible part as well as the radius of gyration of the cluster are taken at this stage. Then the growth resumes, bringing the cluster to a mass $M+\Delta M$, measurements are taken again, and so on until the cluster reached the predetermined size $M_{\max}$. This procedure was repeated for a large enough number of clusters to reduce fluctuations by averaging.

We discuss three runs. In the first two, we use DLA clusters grown on a square lattice with $M_{\max}=200$ particles averaging over 3000 configurations (run No. 1), and with $M_{\max}=20\,000$ particles averaging over 30 configurations (run No. 2). In the third we examine 10 clusters composed of 100 000 particles (run No. 3).

The results are unequivocal: the perimeter, the accessible perimeter, and the inaccessible perimeter all scale as the mass of the cluster. Therefore defining D_P and D_I through

$$M_P \sim R^{D_P} , \tag{5}$$

$$M_I \sim R^{D_I} , \tag{6}$$

in analogy with (3) we find

$$D_P = D_A = D_I = D . \tag{7}$$

In our calculations D is found to range from 1.667 for the smallest clusters to 1.694 for the largest ones. The linear regression analysis of the log-log plots of S, S_A, and S_I vs N, mass of the cluster, gave an exponent practically equal to one in any case for runs No. 2 and No. 3 (Table I and Fig. 3). On the other hand, in run No. 1 $D_P/D=0.93$, $D_A/D=0.81$, and $D_I/D=1.39$. Therefore, if we had limited the analysis to small-size clusters, we could have concluded that $D_A \neq D$. Thus, the problem with obtaining D_A from a multifractal analysis does not stem from the measure of the small probabilities, but from the fact that small clusters have not reached the asymptotic scaling regime. In fact, the ratio

$$\mathcal{R} \equiv \frac{\mathcal{N}_I}{\mathcal{N}_P} \tag{8}$$

(where $\mathcal{N}_I$ and $\mathcal{N}_P$ denote the numbers of inaccessible and perimeter sites, respectively) follows the pattern presented in Fig. 4, rising steeply in the beginning and

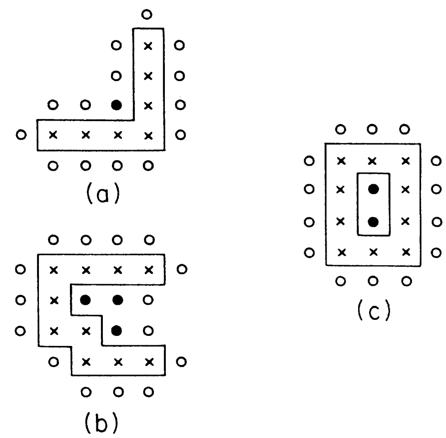

FIG. 2. Some configurations of very small DLA clusters, with explicit perimeter analysis. × denotes DLA sites, ○ denotes accessible perimeter sites, and ● denotes inaccessible perimeter sites.

TABLE I. The various fractal dimensions considered in this work for three different simulations. Here D, D_P, D_A, and D_I denote the fractal dimensions of the cluster itself, its total perimeter, its accessible perimeter, and it inaccessible perimeter.

Run	$M_{\max}$	Config.	D	D_P/D	D_A/D	D_I/D
1	200	3000	1.667	0.928	0.811	1.392
2	20 000	30	1.710	1.000	0.995	1.009
3	100 000	10	1.694	1.000	1.004	0.995

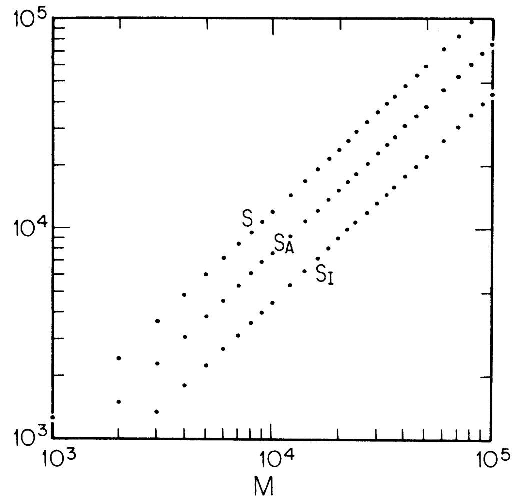

FIG. 3. Log-log plot of the perimeter (S), the accessible perimeter S_A, and the inaccessible perimeter S_I vs the mass of the cluster for run No. 3.

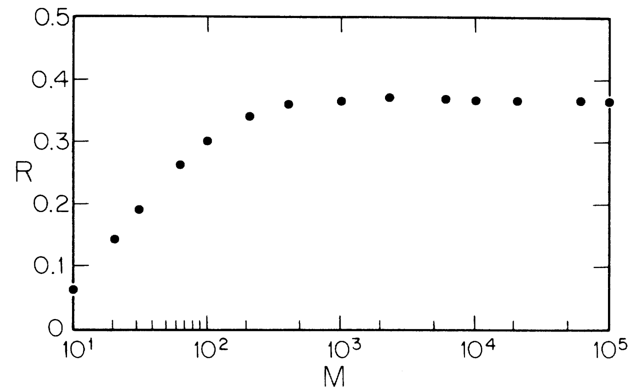

FIG. 4. Log-log plot of the ratio $\mathcal{R} \equiv \mathcal{N}_I/\mathcal{N}_P$ vs the cluster mass.

then reaching a plateau around ~500 particles.

This explains why the two groups that used small clusters[11,13] found values of D_A that were too low. In fact, comparison of our calculations with multifractal calculations on small clusters were carried out, and found to be in remarkably good agreement. The multifractal calculations were obtained by solving the Laplace equation for the surface of the cluster considered as an equipotential[15] to get the equilibrium distribution of surface charges (proportional to the growth probabilities), and then calculating $\lim_{q \to 0^\pm} \tau(q)$ and using (4) to get D_A.

We found that when we calculated the quantity $\{p_i\}_{i \in S}$, the distribution split clearly into two regions, one of which had values of p_i significantly smaller than the resolution of the calculation and hence, presumably correspond to sites with zero growth probability. Their scaling with R was used to determine D_I. Moreover, the number of total perimeter sites was directly measured.

The agreement between the present calculation and the multifractal calculations for small clusters is not only limited to the scaling powers, but extends to the individual points in graphs such as Fig. 3. This shows that the two definitions of accessible perimeter—i.e., sites having $p_i \neq 0$, and sites connected to infinity—do really coincide.

Note that Fig. 4 shows that it is not true that the inaccessible sites are a negligible proportion of the perimeter, destined to become even more negligible as the mass of the cluster increases. In the range we examined, the equilibrium value of the ratio $\mathcal{R}$ is 36.4%.[16] Therefore, at any given time a significant percentage of the perimeter sites are inaccessible, or cannot grow. Nevertheless, the mass of the inaccessible sites scales as the mass of the entire cluster.

For two-dimensional percolation clusters and invasion percolation clusters, the fractal dimension D_A of the accessible perimeter is $\frac{4}{3}$.[17] The fractal dimension D_H of the hull[18] of the percolation cluster is $\frac{7}{4}$ (Ref. 19) and the fractal dimension of the total perimeter D_P is $\frac{91}{48}$ (Ref. 20). The observation of different fractal dimensions associated with the surfaces of percolation clusters provided an additional motivation for the work described in this paper. In general,

$$D_P \geq D_H \geq D_A , \tag{9}$$

while for DLA all three surfaces have the same fractal dimension.[21] We believe that this fact is connected to the fact that loops do *not* occur on all scales in DLA. This is in contrast to percolation, where one finds loops on all scales, and hence, there also occur structures in which a loop is "almost formed." A loop that almost forms leads to a lagoon with a very narrow mouth; the perimeter sites inside the lagoon belong to the hull but not to the accessible perimeter, so the hull scales differently than the accessible perimeter for percolation.

To summarize, we have measured the exponents D_P, D_A, and D_I for DLA clusters, and found them to be equal to the fractal dimension D. Our results also show that the reason why $-\tau(0)$ was observed to be different from D_A lies in the dynamics of the early stages of DLA growth, and not from problems measuring small growth probability. Finally, we determined the equilibrium ratio of inaccessible perimeter sites to surface sites to be 0.365 ± 0.01 for cluster of mass $M \sim 100\,000$.

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