Diffusion of walkers with persistent velocities

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(Received 2 January 1991)

We describe some properties for a phenomenological model of superdiffusion based on a generalization of the persistent random walk in one dimension to continuous time. The time spent moving to either increasing or decreasing x is characterized by a fractal-time pausing time density, $\psi(t) \sim T^{\alpha}/t^{\alpha+1}$, with $1<\alpha<2$. For this system it is shown that asymptotically $p(0,t) \sim 1/t^{1/\alpha}$. The form of the profile is shown to be Gaussian near the peak and to fall off like $tx^{-(1+\alpha)}$ near the tails, and the survival probability is asymptotically proportional to $\exp(-Bt/L^{\alpha})$. These results are confirmed by numerical calculations based on the method of exact enumeration.

I. INTRODUCTION

Considerable recent interest has been focused on the attempt to understand superdiffusive physical systems. The definition of a superdiffusive system is that the displacement of a random walker in the dimension scales with time as a power law, $\langle r^2 \rangle \sim t^{2/d_w}$, where the fractal dimension d_w of the walk is less than, rather than greater than, 2. A number of physical systems exhibit such behavior.[1–6] One example of such a system, originally studied by Matheron and de Marsily[7] was suggested as a model for ground-water transport in stratified media characterized by varying pressure in the direction of the strata (so that the average velocity in each stratum is a random variable). A second example is provided by coherent wave propagation through disordered multiple-scattering media.[8]

Properties of the continuous-time random walk (henceforth CTRW) on a translationally invariant lattice have been used by a number of investigators as a simple way to mimic those of transport in a disordered medium since the pioneering work of Scher and Lax.[9,10] Most models that are based on the CTRW (Refs. 11 and 12) for transport in random media use an assumption that the pausing-time density $\psi(t)$ has a fractal-time behavior, that is

$$\psi(t) \sim T^{\alpha}/t^{1+\alpha} \tag{1}$$

for $t/T \gg 1$, where $0<\alpha<1$, and T is a parameter having the dimensions of time. Such a pausing-time density has no finite integer moments greater than 0, and in consequence, is known to change the probability distribution for the displacement of the random walker at time t.[13]

In this paper we explore a number of properties of the continuous-time generalization[14] of the persistent random walk[12] which incorporates superdiffusion with a continuously variable exponent in contrast to the model in Ref. 7 in which it is shown that $d_w = \frac{4}{3}$. The pausing-time density in this model is characterized by a pausing-time density having a finite first, but infinite second moment. The model for which an expression for $\langle x^2 \rangle$ was given in Ref. 14 consists of a persistent random walk in continuous time for which the pausing-time density has the property given in Eq. (1) with $1<\alpha<2$, and in which the displacement x is related to the time in a single sojourn (that is, moving either right or left) of duration t by

$$f(x,t) = \delta(x \pm vt) , \tag{2}$$

where v is a constant velocity. The further assumption made is that the initial step by the random walker is equally likely to be in the positive or negative direction. Such a model is obviously symmetric, and was shown[14] to have the property that the mean-squared displacement has the asymptotic behavior

$$\langle x^2 \rangle \sim t^{3-\alpha} . \tag{3}$$

For the purpose of studying further properties of the model we simulated a lattice version of the persistent random walk in one dimension in which time is discrete. It is known that the persistent random walk can be regarded as a multistate walk.[15] The model of the present paper is a two-state model, in which the two states are + and −. Our simulated results are for a model more general than that implied by Eq. (2) in that motion in the two states are characterized by a biased diffusive component.

The probability that a single sojourn time in either one of the states in our model is equal to n has the property

$$\psi_n = N/n^{\alpha+1}, \quad n=1,2,3,\ldots \tag{4}$$

where N is a normalizing constant and $1<\alpha<2$. In the + state the random walk takes a step either to the right with probability $p_+ = (1+\epsilon)/2$ or to the left with probability $p_- = (1-\epsilon)/2$, and in the − state these probabilities are reversed. After a sojourn in any given state, the random walker randomly chooses a new state.[16] This differs slightly from the model of Ref. 14, but the asymptotic statistical properties of both models are readily shown to be identical. The model of the persistent random walk that we have just defined reduces to the most elementary walk when $\epsilon=0$ and ψ_n is proportional to a single exponential in n, and for $\epsilon=1$ it corresponds to the random walks studied in Ref. 14. All of the simulations were carried out using the method of exact enumeration.[2] Notice that the model of Ref. 14 corresponds to the choice $\epsilon=1$ in the discrete analog. Our simulations indicate that the scaling relationship in Eq. (3) also remains valid when $0<\epsilon<1$. This can be checked analytically from the theory given in Ref. 14.

We will examine some features of the probability of being at x at step n, $p_n(x)$, for the displacement of such one-dimensional lattice random walks in the limit of long times. We also derived an expression for the survival probability at long times of a random walk between two traps located at $x=\pm L$. Our model allows one to calculate the form of the tails of the curve $p_n(x)$, which are significant for determining the moments of the displacement

The paper is structured as follows. In Sec. II we evaluate $p(x,t)$, the probability density of the displacement for the continuum model in the limit of large time, comparing the analytical results to those obtained by numerical simulation. Section III is devoted to the distribution of first-passage time and ends with a general discussion of the significance of the results.

II. PROBABILITY DENSITY FOR DISPLACEMENT

When $\psi(t)$ has the asymptotic behavior indicated in Eq. (1), its Laplace transform, $\hat{\psi}(s)$, can be expanded in the neighborhood of $s=0$ as

$$\hat{\psi}(s) \sim 1 - sT + (sT_1)^\alpha , \tag{5}$$

where both T and T_1 are constants with the dimensions of time, and $1<\alpha<2$.

Let $p(x,t)$ be the probability density for the displacement of the random walk at time t, conditional on the first step being taken in the positive or negative x direction with probability $\frac{1}{2}$, and let $\hat{p}(\omega,s)$ be the Fourier-Laplace transform of this function, i.e.,

$$\hat{p}(\omega,s) = \int_{-\infty}^{+\infty} e^{iwx}dx \int_0^\infty e^{-st}p(x,t)dt . \tag{6}$$

To write the expression for $\hat{p}(\omega,s)$ in a compact fashion we let $z=s+iv\omega$, and $\bar{z}=s-iv\omega$, where both s and ω are to be regarded as real variables and the parameter v is defined in Eq. (2). An exact representation of $\hat{p}(\omega,s)$ for the particular model under consideration has been shown to be[14,17]

$$\hat{p}(\omega,s) = \frac{1}{[1-\hat{\psi}(z)\hat{\psi}(\bar{z})]}\mathrm{Re}\left\{\frac{[1-\hat{\psi}(z)][1+\hat{\psi}(\bar{z})]}{z}\right\} , \tag{7}$$

where Re means "real part" of the function in parentheses.

Let us first analyze the properties of $\hat{p}(\omega,s)$ in the regime in which both ω and s are small, which allow us to expand $\hat{\psi}(z)$ using Eq. (5). This regime corresponds, in the time-space domain, to having both (t/T) and x^2 be large. Let λ be the constant $\lambda = T_1^\alpha/T$. A straightforward but tedious expansion of Eq. (7) to the lowest two orders of z yields

$$\hat{p}(\omega,s) \sim \frac{1}{s}\left\{1-\frac{\lambda}{2}(z^{\alpha-1}+\bar{z}^{\alpha-1})+\frac{\lambda}{2s}(z^\alpha+\bar{z}^\alpha)+\cdots\right\} . \tag{8}$$

Since we can represent z^α in the form of an integral as

$$z^\alpha = \frac{z}{z^{1-\alpha}} = \frac{z}{\Gamma(1-\alpha)}\int_0^\infty \xi^{-\alpha}e^{-z\xi}d\xi , \tag{9}$$

it follows that one can replace Eq. (8) by an integral representation, using the identities

$$z^{\alpha-1}+\bar{z}^{\alpha-1} = \frac{2}{\Gamma(2-\alpha)}\int_0^\infty \frac{d\xi}{\xi^{\alpha-1}}[s\cos(\omega v\xi) + \omega v \sin(\omega v \xi)]e^{-s\xi} ,$$
$$\frac{1}{2s}(z^\alpha+\bar{z}^\alpha) = \frac{1}{s\Gamma(2-\alpha)} \times \int_0^\infty \frac{d\xi}{\xi^{\alpha-1}}[(s^2-w^2v^2)\cos(\omega v\xi) + 2s\omega v\sin(\omega v\xi)]e^{-s\xi} . \tag{10}$$

Thus Eq. (8) is seen to be expressible in terms of the integrals

$$C_\beta = \int_0^\infty e^{-s\xi}\frac{\cos(\omega v\xi)}{\xi^\beta}d\xi ,$$
$$S_\beta = \int_0^\infty e^{-s\xi}\frac{\sin(\omega v\xi)}{\xi^\beta}d\xi . \tag{11}$$

The combination of Eqs. (8), (10), and (11) allows us to rewrite Eq. (8) as

$$\hat{p}(\omega,s) \sim \left\{\frac{1}{s} - \frac{\lambda}{\Gamma(2-\alpha)}\left[\left[\frac{\omega v}{s}\right]^2 C_{\alpha-1} - \frac{\omega v}{s}S_{\alpha-1}\right]\right\} . \tag{12}$$

The expression in this last equation constitutes a starting point for finding an approximation to two quantities of interest, namely the probability density for remaining at the origin, $p(0,t)$, and the probability density $p(x,t)$ for large x, defined by a suitable scaling limit. To find the first of the two quantities we note that the Laplace transform of $p(0,t)$, which will be denoted by $\bar{p}(0,s)$, can be found from $\hat{p}(\omega,s)$ by

$$\bar{p}(0,s)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{p}(\omega,s)d\omega \; . \tag{13}$$

We do not know $\hat{p}(\omega,s)$ exactly, but we do have an approximation to it in Eq. (12). The form of that equation does not allow us to perform the integration over ω indicated in Eq. (13). However, the joint assumption that Eq. (12) is the lowest-order term in the expansion of an exponential and that the principal contribution to the value of the integral comes from the neighborhood of $\omega=0$ allows us to insert the approximation

$$\hat{p}(\omega,s)\sim\frac{1}{s}\exp\left[-\frac{\lambda}{\Gamma(2-\alpha)}\left[c_1\frac{(\omega v)^{\alpha}}{s}-c_2(\omega v)^{\alpha-1}\right]\right] \tag{14}$$

(where c_1,c_2 are constants of no relevance to our argument) into Eq. (13). The resulting integral is a convergent one. The integral in Eq. (13) can be evaluated by transforming the variable ω to a new variable ρ, by setting $\omega=\rho s^{1/\alpha}/v$. In this way we find

$$\begin{aligned}\hat{p}(0,s)&\sim\frac{1}{2\pi s^{1-1/\alpha}}\int_{-\infty}^{\infty}\exp(-A\rho^{\alpha}+B\rho^{\alpha-1}s^{(\alpha-1)/\alpha})d\rho\\ &\sim\frac{1}{2\pi s^{1-1/\alpha}}\int_{-\infty}^{\infty}\exp(-A\rho^{\alpha})d\rho \; ,\end{aligned} \tag{15}$$

where A is the constant $\lambda c_1/\Gamma(2-\alpha)$, and we have dropped a term consistent with the limit $s\to 0$. Since the integral in the second line is a constant, we can make use of a Tauberian theorem for Laplace transforms[18] to infer that

$$p(0,t)\sim\frac{1}{t^{1/\alpha}}, \quad t\to\infty \; . \tag{16}$$

This differs from the behavior suggested by the often-made assumption of transport in a disordered medium, that the asymptotic behavior of $p(0,t)$ is related to $\langle x^2\rangle$ by

$$p(0,t)\sim C/\langle x^2\rangle^{1/2} \; , \tag{17}$$

where C is a constant. The result so obtained for the asymptotic behavior of $p(0,t)$ was checked numerically (cf. Fig. 1) because of the nature of the many approximations in finding it.

One can calculate the shape of the curve around $x=0$ by starting from the Laplace transform $\bar{p}(x,s)$. The integral representation of $\bar{p}(x,s)$ is obtained from the Fourier transform in Eq. (13) as

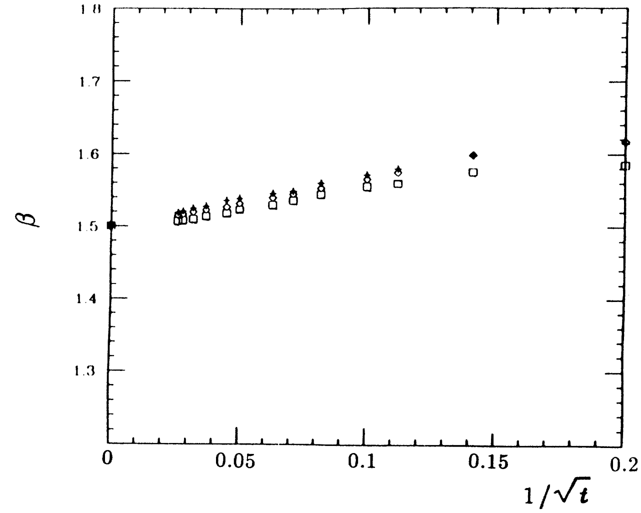

FIG. 1. Asymptotic behavior of the exponent of time in $p(0,t)$ for $\alpha=1.5$ and $\epsilon=0.5$ (□), 0.8 (◇), and 1.0 (+). Results of the exact enumeration calculation were then fit to the form $p(0,t)=A/t^{1/\beta}$, A being a constant. The convergence of β to the true value α as a function of t is indicated in the figure.

$$\begin{aligned}\bar{p}(x,s)&=\frac{1}{\pi s}\int_0^{\infty}\exp\left[-A\frac{(\omega v)^{\alpha}}{s}\right]\cos(\omega v)d\omega\\ &\sim\frac{1}{\pi s}\int_0^{\infty}\exp\left[-A\frac{(\omega v)^{\alpha}}{s}\right]\\ &\quad\times\left[1-\frac{(\omega v)^2}{2}+\cdots\right]d\omega \; .\end{aligned} \tag{18}$$

The integrals are readily evaluated and, by means of a Tauberian theorem, lead to the following approximate Gaussian form for $p(x,t)$ in the neighborhood of the origin

$$\begin{aligned}p(x,t)&\sim\frac{K}{t^{1/\alpha}}\left[1-K'\left[\frac{x^2}{t^{2/\alpha}}\right]\right]\\ &\sim\frac{K}{t^{1/\alpha}}\exp\left[-K'\left[\frac{x^2}{t^{2/\alpha}}\right]\right] \; ,\end{aligned} \tag{19}$$

where K and K' are readily calculated constants.

Numerical simulations on the discrete system strongly support the asymptotic expression for the time dependence in Eq. (16) for $p_n(0)$, not only for the value of $\epsilon=1$ in the transition probabilities but also for $\epsilon<1$. Figure 1 shows the asymptotic behavior of the exponent n in $p_n(0)$ for $\alpha=1.5$ and several values of ϵ: 0.5, 0.8, and 1.0. These support the assertion that this exponent is independent of the value of ϵ.

We can make use of the approximation in Eq. (12) to predict the form of the scaling behavior of $p(x,t)$ at long times. When we insert the expressions for the integrals $C_{\alpha-1}$ and $S_{\alpha-1}$ explicitly in this equation,

$$\begin{aligned}\hat{p}(\omega,s) \sim & \frac{1}{s} - \frac{\lambda(\omega v)^2}{s^2\Gamma(2-\alpha)}\int_0^\infty e^{-s\xi}\frac{\cos(\omega v\xi)}{\xi^{\alpha-1}}d\xi \\ & + \frac{\lambda(\omega v)}{s\Gamma(2-\alpha)}\int_0^\infty e^{-s\xi}\frac{\sin(\omega v\xi)}{\xi^{\alpha-1}}d\xi \\ = & \frac{1}{s} - \frac{\lambda(\omega v)^\alpha}{s^2\Gamma(2-\alpha)}\int_0^\infty \exp\left[-\frac{sz}{\omega v}\right]\frac{\cos z}{z^{\alpha-1}}dz \\ & + \frac{\lambda(\omega v)^{\alpha-1}}{s\Gamma(2-\alpha)}\int_0^\infty \exp\left[-\frac{sz}{\omega v}\right]\frac{\sin z}{z^{\alpha-1}}dz \ . \end{aligned} \tag{20}$$

It will always be assumed that both s and ω are positive. Since we are interested in the behavior of the tails of $p(x,t)$ for large but fixed values of t, we will work in the limits $\omega, s \to 0$ together with the assumption that

$$\lim_{\omega,s\to 0} \omega/s \to \text{const} \ . \tag{21}$$

In these limits the last two terms on the right-hand side of Eq. (20) are of the same order of magnitude, i.e., they are $O((\omega v)^\alpha/s^2)$. But because of the condition imposed in Eq. (21) we can assert that

$$(\omega v)^\alpha \ll s \ . \tag{22}$$

This allows us to approximate $\hat{p}(\omega,s)$ by

$$\begin{aligned}\hat{p}(\omega,s) & \sim \frac{1}{s} - A\frac{(\omega v)^\alpha}{s^2} \\ & \sim \frac{1}{s}\left[1 - A\frac{(\omega v)^\alpha}{s}\right] \\ & \sim \frac{1}{s}\left[\frac{1}{1+A\frac{(\omega v)^\alpha}{s}}\right] \\ & = \frac{1}{s+A(\omega v)^\alpha} \ , \end{aligned} \tag{23}$$

where A is the constant

$$A = \frac{\lambda}{\Gamma(2-\alpha)}\int_0^\infty \exp\left[-\frac{sz}{\omega v}\right]\left[\frac{\cos z - \frac{s}{\omega v}\sin z}{z^{\alpha-1}}\right]dz \ . \tag{24}$$

It is shown in the Appendix that A is indeed positive as required for our derivation of results.

One readily inverts the Laplace transform in Eq. (23) to show that

$$\bar{p}(\omega,t) \sim \exp[-A(\omega v)^\alpha t] \ , \tag{25}$$

valid at large t. The inverse transform of this function can be expressed as

$$p(x,t) \sim \frac{1}{\pi v t^{1/\alpha}}\int_0^\infty \exp(-u^\alpha)\cos\left[\frac{xu}{vt^{1/\alpha}}\right]du \ , \tag{26}$$

which can be expanded in an asymptotically convergent series, of which the lowest-order term is[18]

$$p(x,t) \sim \frac{1}{\pi}\Gamma(1+\alpha)\sin\left[\frac{\pi\alpha}{2}\right]v^\alpha\frac{t}{x^{(1+\alpha)}} \ . \tag{27}$$

Numerical simulations support this asymptotic behavior for sufficiently large x [but note that when x^2 exceeds $(vt)^2$, $p(x,t)$ must be identically equal to 0]. Figure 2(a) shows a plot of $p(x,t)$ for $\epsilon=1$ and $\alpha=1.5$. Based on Eqs. (19) and (27) we can roughly summarize the expressions for $p(x,t)$ as

$$p(x,t) \sim \begin{cases} \frac{1}{t^{1/\alpha}}, & x < t^{1/\alpha} \\ \frac{1}{t^{1/\alpha}}\left[\frac{x}{t^{1/\alpha}}\right]^{-(1+\alpha)}, & t^{1/\alpha} < x < t \\ 0, & x > t \ . \end{cases} \tag{28}$$

Figure 3 shows a plot of $p(x,t)t^{1/\alpha}$ as a function of the scaled variable $x/t^{1/\alpha}$ for $\alpha=1.5$ and several values of t: 200, 400, 600, 800. As it can be seen the collapse of the data on a single curve supports the result in Eq. (28). Also this form is consistent with the requirement that

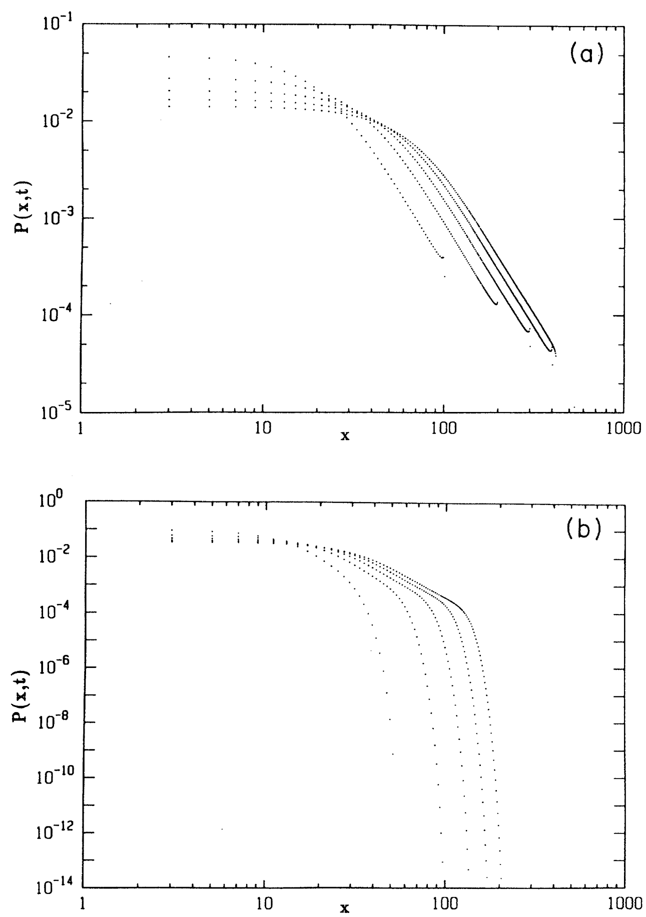

FIG. 2. (a) Log-log plot of probability density $p(x,t)$ for $\epsilon=1$ and $\alpha=0.5$. (b) Log-log plot of probability density $p(x,t)$ for $\epsilon=0.5$ and $\alpha=1.5$. From the top, the curves are for $t=100$, 200, 300, 400, and 500.

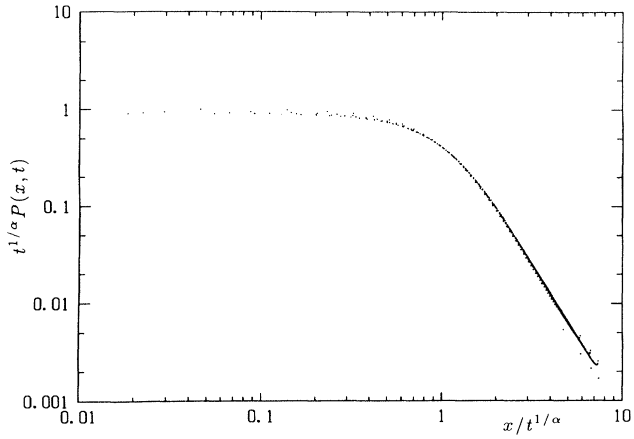

FIG. 3. Log-log plot of $t^{1/\alpha}p(x,t)$ as a function of the scaled variable $x/t^{1/\alpha}$ for $\alpha=1.5$ and several values of t: 200, 400, 600, and 800.

$p(x,t)$ be normalized. We can write approximately

$$\int_0^{t^{1/\alpha}} \frac{1}{t^{1/\alpha}} dx + \int_{t^{1/\alpha}}^{t} \frac{1}{t^{1/\alpha}} \left[\frac{x}{t^{1/\alpha}} \right]^{-(1+\alpha)} dx \approx 1 \ . \tag{29}$$

It is also readily verified that the approximate form in Eq. (28) implies the asymptotic behavior of $\langle x^2 \rangle$ given in Eq. (3).

When $\epsilon<1$, the probability density $p(x,t)$ clearly exhibits an extra regime which appears in ordinary diffusion. Figure 2(b) shows a plot of $p(x,t)$ for $\epsilon=0.5$ and $\alpha=1.5$. The extra diffusive regime is obviously to be expected since there are now two sources of variability, that inherent in $\psi(t)$, in addition to fluctuations in the direction of the steps. The width of this extra region is asymptotically proportional to $t^{1/2}$. It should be noted that the actual slope in the linear region of the log-log plot of $p(x,t)$ in Figs. 2(a) and 2(b) is in fact about 20% greater than $1+\alpha$. This is probably due to not having reached the asymptotic regime.

III. DISTRIBUTION OF FIRST-PASSAGE TIME

In this section we present an argument suggesting the form of the distribution of first-passage time $\langle t \rangle$ for a random walker between two traps separated by a distance L. A formalism for calculating the survival probability of such a random walker is given in Ref. 19, but we present a more heuristic derivation of this quantity leading to results in agreement with our simulations. Consider a line of length L, in the limit $L\to\infty$. In order for the random walker to exit the line, it must at some time make a step whose length is the order of magnitude of L. When the velocity is constant, as in the present case, the probability that the displacement in a single sojourn is greater than L/v is $\Psi(L/v)$, where

$$\Psi(t) \equiv \int_t^{\infty} \psi(\tau) d\tau \tag{30}$$

is the probability that a single sojourn last longer that a time t. This probability behaves asymptotically as

$$\Psi(t) \sim \frac{1}{\alpha} \left[\frac{T}{t} \right]^{\alpha} . \tag{31}$$

The probability that n steps are made before one is made that is as large as L/v is approximately

$$\theta_n \sim \left[1-\Psi\left[\frac{L}{v}\right] \right]^n \Psi\left[\frac{L}{v}\right] . \tag{32}$$

We next connect the value of the discrete parameter n to the continuous time t by using a relation suggested by renewal-theoretic considerations, $n\sim t/\langle t \rangle$, which converts Eq. (32) to

$$\theta(t) \sim \left[\frac{vT}{L}\right]^{\alpha} \left[1-\left[\frac{vT}{L}\right]^{\alpha} \right]^{t/\langle t \rangle} \sim \left[\frac{vT}{L}\right]^{\alpha} \exp\left[-\frac{(vT)^{\alpha}}{\langle t \rangle} \frac{t}{L^{\alpha}} \right] . \tag{33}$$

Hence the survival probability is

$$S(t) \sim \exp\left[-\frac{(vT)^{\alpha}}{\langle t \rangle} \frac{t}{L^{\alpha}} \right] . \tag{34}$$

If this prediction is taken seriously, the mean trapping time should scale with the length as L^{α}. In Fig. 4(b), the mean trapping time is plotted as a function of L in a log-log scale. The plotted points indicate a slope in agreement with this prediction.

As a final point we consider the trapping problem in one-dimension with the two-state random walk considered in this paper. The trapping problem is defined in terms of an infinite line, for which the probability that a given interval $(x,x+dx)$ contains a trap is equal to cdx, where c is the trap concentration. One is required to find the survival probability of a walker averaged over all values of length separating adjacents traps. Equation (34) gives the survival probability conditional on such a length being equal to L. The survival probability averaged over all intervals is then

$$\langle S(t) \rangle \equiv c^2 \int_0^{\infty} L S(t|L) e^{-cL} dL \ . \tag{35}$$

On the assumption that the time is large, an approximation to the value of $\langle S(t) \rangle$ may be found through the use of the method of steepest descents, leading to the estimate of the leading term in $\ln\langle S(t) \rangle$:

$$\ln\langle S(t) \rangle \sim -K c^{\alpha/(1+\alpha)} t^{1/(1+\alpha)} , \tag{36}$$

where K is a readily determined constant. When $\alpha=2$ the time dependence of this result coincides with the one-dimensional result of Donsker and Varadhan[20] and when α decreases, longer steps are taken by the walker, and therefore the survival probability decreases.

IV. DISCUSSION

(1) The probability density for being at the origin $p(0,t)$, whose asymptotic form is given in Eq. (16) decreases more slowly than the corresponding density for random walks on a translationally invariant line or for

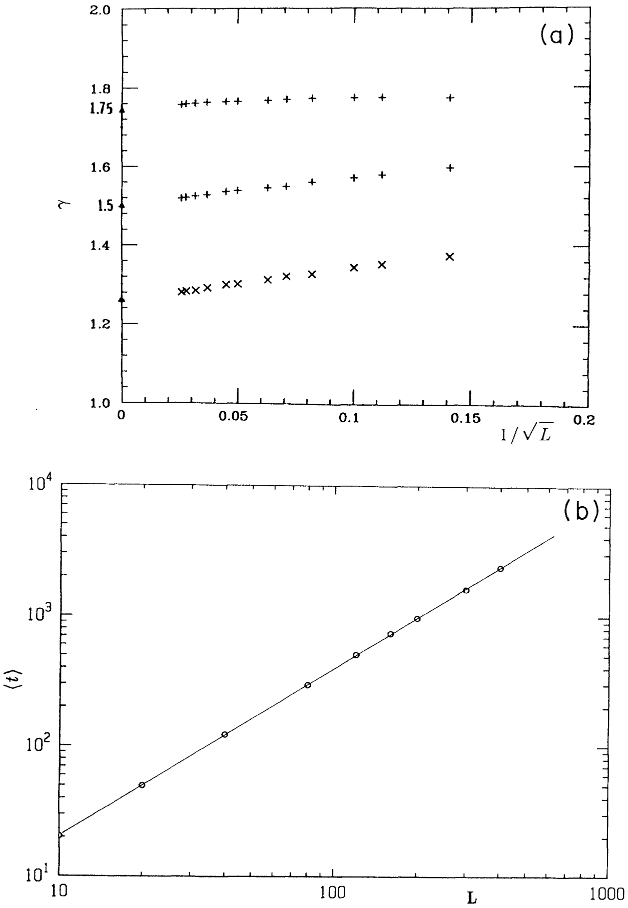

FIG. 4. (a) Asymptotic behavior of the length exponent γ in the survival probability which is fitted to a form $S_L(t) \sim \exp(-at/L^{\gamma})$ plotted as a function of $1/L^{1/2}$. Our data indicate the convergence of γ to the conjectured value $\gamma = \alpha$. From the top, the curves are for the values $\alpha = 1.75$, 1.5, and 1.25. (b) Average time for the walker to get trapped for $\alpha = 1.25$.

random walks on fractals. This result stems from the fact that the probability density has no characteristic length scale, which leads to a failure of Eq. (17).

(2) The present model introduces long-range correlations between the steps of the walker characterized by a two-point velocity correlation function with a power-law behavior $C(j) \sim j^{1-\alpha}$, as suggested by the plot in Fig. 5 for the case $\alpha = 1.5$ and $n = 3000$. It is interesting to note that Peng *et al.*[21] have studied a random walk in which the long-range correlations are of the same form, but in which $p(x,t)$ is Gaussian over the entire range of x, in contrast with our present results in which the Gaussian form only holds in the neighborhood of the origin.

After this work was completed, we learned of similar

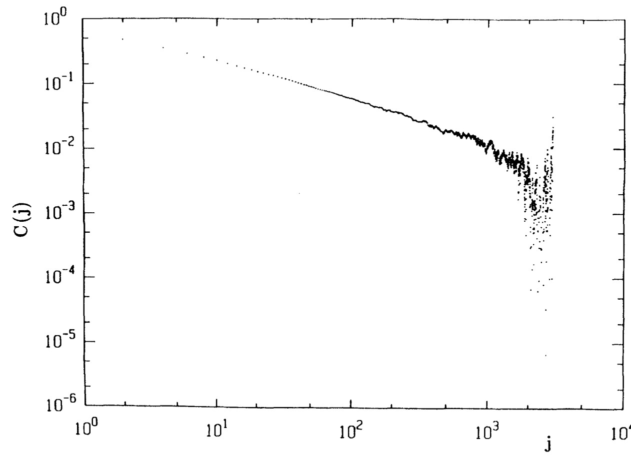

FIG. 5. Two-point velocity correlation $\langle v(0)v(j) \rangle = C(j)$ as a function of j for $\alpha = 1.5$ and $n = 3000$. Here $v(n)$ is the velocity of the random walker at step n.

results obtained for the case $\epsilon = 1$ using quite different analytical and numerical methods.[22]

ACKNOWLEDGMENTS

We wish to thank C.-K. Peng, S. Redner, and M. Schwartz for helpful discussions. The Center for Polymer Studies is supported by grants from the Office of Naval Research and National Science Foundation. One of us (M.A.) acknowledges the support of Intevep SA, while S.H. thanks the U.S.–Israel Binational Science Foundation for partial support.

APPENDIX: PROOF OF THE POSITIVITY OF A DEFINED IN EQ. (24)

Let β be the constant $s/(\omega v)$. In Eq. (24) we use the representation

$$\frac{1}{z^{\alpha-1}} = \frac{1}{\Gamma(\alpha-1)} \int_0^{\infty} \xi^{\alpha-2} e^{-\xi x} d\xi \qquad \text{(A1)}$$

and interchange the orders of integration with respect to ξ and z, carrying out the integration with respect to z first. This leads to the alternate representation

$$A = \frac{\lambda}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \times \int_0^{\infty} \xi^{\alpha-2} d\xi \int_0^{\infty} e^{-(\beta+\xi)x} (\cos z - \beta \sin z) dz = \frac{\lambda}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \int_0^{\infty} \frac{\xi^{\alpha-1}}{1+(\beta+\xi)^2} d\xi\,, \qquad \text{(A2)}$$

which is manifestly positive.

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